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The homogeneous chaos from the standpoint of vector measures

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The homogeneous chaos from the standpoint of vector measures

Dedicated to the memory of Norbert Wiener, 1894–1964

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Norbert Wiener's 1938 theory of the homogeneous chaos involves measures ξ_p over

$\mathbb{R}^p$ with values in the Hilbert space $\mathcal{L}_2 := L_2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R})$, these values being non-Gaussian random variables with complicated covariance structure.

In the current treatments, due to Kakutani, Ito, Segal, Gross and their followers, an important part of this theory is recovered, even though the measure ξ_p itself and the difficult questions of its countable additivity, semi-variation, covariance, integration, etc., are bypassed. Missed in such an approach, however, are interesting, difficult and highly combinatorial results, the embryonic forms of which appear in Wiener's later writings.

In this paper we face the measure ξ_p head on, adhering strictly to the canons of the Lebesgue–Pettis theory of Banach-space valued measures and their integration. A fundamental clue is provided by the equality (75) in Wiener's paper, which is conspicuously missing in the treatments in vogue. Our treatment is coordinate-free: bases are introduced only in the last section in order to establish a nexus with the important work of Cameron & Martin, from which the cited authors take off.

An unexpected bonus is that a combinatorial result, pertaining to integrability with respect to the measure ξ_p, completely resolves the difficult problem of liftings that has appeared in the Hu–Meyer treatment of the Feynman integral.

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1. Introduction

(*a*) *On infinite-dimensional vector-valued measures*

The subject of infinite-dimensional vector-valued measures has gained substantially from two path-breaking contributions of Norbert Wiener. In the first of these, Wiener (1923), he introduced a probability measure on the space of paths of an idealized version of Einstein's Brownian motion process, and thereby made possible the demarcation of a countably additive measure ξ with values in the Hilbert space $L_2[0,1]$, with the potent property that its total variation measure[1] $|\xi|$ has exactly the values 0 and ∞, i.e.

(1) $$\text{Range}\,|\xi| = \{0, \infty\}.$$

This ξ is obtained by extension, starting from its definition for intervals, which is

(2) $$\forall a, b \in \mathbb{R} \quad \& \quad \forall \alpha \in [0,1], \qquad \xi(a,b](\alpha) := x(b,\alpha) - x(a,\alpha),$$

where $\{x(t,\alpha) : t \in \mathbb{R} \text{ and } \alpha \in [0,1]\}$ is Wiener's idealized Brownian motion stochastic process (SP) over the probability space [0,1] with Lebesgue measure.

In the early 1930s the integration of scalar functions ϕ with respect to this ξ was defined by Paley & Wiener (1934, pp. 151–156) by a judicious recourse to integration by parts, defined more directly by Doob (1953, pp. 426–429), and following Doob's footsteps for all ξ with values in an arbitrary Hilbert space $\mathcal{H}$, which obey the requirement

(3) $$(\xi(A), \xi(B))_{\mathcal{H}} = \mu(A \cap B) \geqslant 0$$

by this writer (Masani 1968). For Wiener's ξ, $\mathcal{H}$ is $L_2[0,1]$ and μ is the Lebesgue measure ℓ_1 over $\mathbb{R}$.

The more general integration of ϕ with respect to a measure ξ with values in a

[1] The total variation measure $|\xi|$ is defined exactly as for complex-valued measures, except for using the $L_2[0,1]$ norm instead of the absolute value.

Banach space $\mathcal{X}$, was begun by N. Dunford and his colleagues in the mid-1950s, was simplified and systematized in the 1970s and 1980s in several papers by D. R. Lewis, E. G. F. Thomas, J. K. Brooks and N. Dinculeanu and by H. Niemi and the writer. Of these papers we need only mention the ones we will use, namely Brooks (1971), Brooks & Dinculeanu (1974) and Masani & Niemi (1989a, b, 1992, cited as [MN,I,II,III] in the sequel). A Fubini theorem for the product measure $\xi \times \mu$, where μ is scalar-valued, shows that the completion of the theory requires consideration of the integration of vector-valued functions $\vec{f}$ with respect to the scalar measure μ [MN,III,9.7]. Such integration had been broached much earlier in the 1930s and 1940s by S. Bochner, J. R. Pettis, N. Dunford, G. Birkhoff, R. S. Phillips, and their followers in several important papers.

We could dispense with integrals of the type $\int_{\mathbb{R}} \phi(t)\xi(\mathrm{d}t)$ in favour of those of the type $\int_{\mathbb{R}} \vec{f}(t)\mu(\mathrm{d}t)$, were a substitution principle such as

$$\int_{\mathbb{R}} \phi(t)\xi(\mathrm{d}t) = \int_{\mathbb{R}} \phi(t)\frac{\mathrm{d}\xi}{\mathrm{d}|\xi|}(t)|\xi|(\mathrm{d}t)$$

available. But such a principle is ruled out by the stipulation (1). Thus *for all ξ subject to (1), the integration $\int_{\mathbb{R}} \phi(t)\xi(\mathrm{d}t)$ is intrinsic.* Furthermore, many such ξ admit the implication

$$\int_{\mathbb{R}} \phi(t)\xi(\mathrm{d}t) = \int_{\mathbb{R}} \psi(t)\xi(\mathrm{d}t) \Longrightarrow \phi(t) = \psi(t) \quad \text{a.e.},$$

reminiscent of the familiar condition for linear independence of a sequence of vectors $(x_1, \ldots, x_n)$, namely,

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} b_k x_k \Longrightarrow \forall k = 1, 2, \ldots, n, \quad a_k = b_k.$$

Thus we ought to look upon many vector measures ξ, subject to (1), as 'continuous' basis for the Hilbert or Banach space in question, made up of 'infinitesimal' vectors $\xi(\mathrm{d}t)$, and should look upon equalities such as

$$x = \int_{\mathbb{R}} \phi(t)\xi(\mathrm{d}t)$$

as 'expansions' of the vector x in terms of such 'measure basis'.

In Wiener (1938, subsequently cited as [W]), came his second path-breaking contribution to the field. He introduced the p-fold product ξ_p of his original measure given in (2), and defined for $A = (a_1, b_1] \times \cdots \times (a_p, b_p]$ by the product,

$$\xi_p(A) = \xi(a_1, b_1] \cdot \cdots \cdot \xi(a_1, b_1]. \tag{4}$$

The p factors on the RHS are normally distributed random variables with zero means, which are not necessarily independent. It is easily seen that while $\xi_p(A)$ is in $L_2[0, 1]$, it is not normally distributed and not subject to a condition of the type (3).

That the study of ξ_p transcends the vector measure theory currently in existence is clear from the fact that this theory is confined to measures with values in a vector space and not in a linear algebra, whereas for the 1938 Wiener measure ξ_p, it is the multiplication of vectors that is crucial. Moreover, an extension of the current theory of Banach space valued measures to Banach algebra valued measures will not meet the demands of Wiener's measure ξ_p, since the $\mathcal{L}_2$-norm, relevant to its theory, is not a Banach algebra norm; indeed for non-independent normally distributed random variables X, Y, we find that $|X \cdot Y|_{\mathcal{L}_2} > |X|_{\mathcal{L}_2} \cdot |Y|_{\mathcal{L}_2}$. Here $\mathcal{L}_2 = L_2[0,1]$. (This is a trivial consequence of the lemma 2.1 on normal variates.)

Had Wiener taken the easy way, and defined $\xi_p(A)$ not by (4), but by the corresponding tensor product

$$\xi_p(A) = \xi(a_1, b_1] \otimes \cdots \otimes \xi(a_p, b_p],$$

all these difficulties would have vanished. For this ξ_p has values in the Hilbert space $\{L_2[0,1]\}^{\otimes p} \simeq L_2\{[0,1]^p\}$, and obeys the simple equality (3) in the form,

$$(\xi_p(A), \xi_p(B))_{L_2\{[0,1]^p\}} = \ell_p(A \cap B),$$

where ℓ_p is the Lebesgue measure over $\mathbb{R}^p$. But such an escape into Fock space would have left out the solid crust of Wiener's theory which rests on sticking to just one Hilbert space, $L_2[0,1]$, for all $p \in \mathbb{N}_+$.

(*b*) *The purpose of this paper*

Our objective is to deal systematically and rigorously with the p-fold product measures $\xi_p = \xi \times \xi \times \cdots \times \xi$ appearing on the left side of (4) and their integration, by adhering strictly to the Lebesgue pattern outlined in the definitions (A.1), (A.9), (A.10), (A.12), (A.14), (A.25) and (A.26) of Appendix A. In this a measure ρ with values in a topological vector space $\mathcal{X}$ comes first, a $[0,\infty]$-valued ρ-norm $|f|_{1,\rho}$ of measurable functions f comes next, then the class $\mathcal{G}_{1,\rho}$ of ρ-integrable functions is defined by the condition $|f|_{1,\rho} < \infty$, the class[2] $\mathcal{P}_{1,\rho}$ emerging as the closure of the simple functions, and finally comes integration, defined as a linear operator $\mathbb{E}_\rho$ on the vector space $\mathcal{P}_{1,\rho}$ to the original vector space $\mathcal{X}$. In our case, $\mathcal{X}$ is a Hilbert space, and $\mathcal{P}_{1,\rho} = \mathcal{G}_{1,\rho}$.

We have to cope with the new questions that arise from the inhering multiplication. One such, which demands early attention, is the determination of the inner product $(\xi_p(A), \xi_q(B))_{\mathcal{L}_2}$ in terms of familiar scalar measures, in the spirit of the equation (3). This, and corresponding questions for integrals, are beset by severe combinatorial complexities, to handle which we have had to work out a scheme of combinatorial concepts. This has contributed to the extreme length and difficulty of this paper.[3]

(*c*) *Basic notation used in the paper*

In order to demarcate the issues involved, and describe clearly the new results, we must now prescribe the basic notation to be used.

1.1. *Basic Notation.*

(*a*) The symbols $\forall$ and $\exists$ stand for the universal and existential quantifiers. LHS and RHS abbreviate left-hand side and right-hand side, respectively. The symbol := means equal by definition. For any set A, $\#(A)$ denotes the cardinality of A, and χ_A the indicator function of A. $\mathrm{Rstr.}_A\, f$ stands for the restriction of the function f to a subset A of its domain. The symbol $\|$ means disjoint.

(*b*) $\mathbb{F}$ refers to either the real or complex number fields $\mathbb{R}$ or $\mathbb{C}$, and $\mathbb{N}$ to the set of

[2] $\mathcal{G}$, $\mathcal{P}$ in honour of Gelfand and Pettis, since functionals in $\mathcal{X}'$, the dual of $\mathcal{X}$, which they use, play an intrinsic role in yielding the norm $|f|_{1,\rho}$, apart from which the theory is Lebesgue in spirit.

[3] The writer is most grateful to the Royal Society for accepting its publication, and to Professor C. R. Rao, F.R.S., for communicating it, and to the referee for his very careful comments. He regrets not having the paper ready for publication in 1994, the centenary of Wiener's birth.

all integers. $\mathbb{R}_+$, $\mathbb{N}_+$ and $\mathbb{R}_{0+}$, $\mathbb{N}_{0+}$ denote the subsets of positive elements and the subsets of non-negative elements of $\mathbb{R}$ and $\mathbb{N}$, respectively.

(*c*) The symbols $(a, b]$, $[a, b]$, etc., where $a, b \in \mathbb{R}$ and $a \leqslant b$, denote the half-open closed, closed, etc., intervals of $\mathbb{R}$. However, when $m, n \in \mathbb{N}$ and $m \leqslant n$, we shall write $[m, n]$ for the set $\{m, m+1, m+2, \ldots, n\} \subseteq \mathbb{N}$.

(*d*) For $p \in \mathbb{N}_+$, the space $\mathbb{R}^p$ is defined by

$$\mathbb{R}^p := \mathbb{R}^{[1,p]} := \{x : x \text{ is a function on } [1, p] \text{ to } \mathbb{R}\}.$$

We let

$$\mathbb{R}^0 := \{0\} \subset \mathbb{R}.$$

The symbol $(a, b, \ldots, \ell)$ with 12 terms, where $a, b, \ldots, \ell \in \mathbb{R}$, will stand for the function $x \in \mathbb{R}^{12}$ such that

$$x(1) = a, x(2) = b, \ldots, x(12) = \ell.$$

(*e*) For $\emptyset \neq \mathcal{F} \subseteq 2^X$ and $\emptyset \neq \mathcal{G} \subseteq 2^Y$, $\mathcal{M}(\mathcal{F}, \mathcal{G})$ is the set of all f in Y^X, which are $\mathcal{F}$, $\mathcal{G}$ measurable, i.e. $f^{-1}(G) \in \mathcal{F}$ for each $G \in \mathcal{G}$. $\mathcal{S}(\mathcal{F}, \mathbb{R})$ is the class of all $\mathbb{R}$-valued $\mathcal{F}$ simple functions on X.

(*f*) For topological vector spaces X and Y, and $A \subseteq X$, $\langle A \rangle$ is the linear manifold spanned by A in X, and $\mathfrak{S}(A) := \operatorname{cls}\langle A \rangle$ where 'cls' stands for 'the closure of'. $\mathrm{L}(X, Y)$ is the class of all linear operators on X into Y, and $\mathrm{CL}(X, Y)$ is the set of continuous linear operators on X into Y.

(*g*) For $Y_0 \subseteq Y$ a vector space, and $\mathcal{R}$ a ring of subsets of a set Ω, $\mathrm{FA}(\mathcal{R}, Y_0)$ and $\mathrm{CA}(\mathcal{R}, Y_0)$ stand for the sets of all finitely additive and of all countably additive measures ξ on $\mathcal{R}$ with values in Y_0. The symbols 'FA' and 'CA' abbreviate 'finitely additive' and 'countably additive'. $\mathcal{M}_\xi := \langle \operatorname{Range} \xi \rangle$ and $\mathcal{S}_\xi := \mathfrak{S}\{\operatorname{Range} \xi\}$. (Thus $\mathcal{M} \subseteq \operatorname{cls} \mathcal{M}_\xi = \mathcal{S}_\xi \subseteq \operatorname{cls} Y_0$.) $\sigma\text{-alg}(\mathcal{F})$ is the σ-algebra generated by the family $\mathcal{F}$ of subsets of a set X; likewise for a $\sigma\text{-ring}(\mathcal{F})$, $\delta\text{-ring}(\mathcal{F})$, etc.

(*h*) For any measure ξ on a set-family $\mathcal{F}$ over a space Λ, and any integrable function f on Λ, we shall write $\mathbb{E}_\xi(f)$ for the (Lebesgue or Lebesgue–Pettis) integral $\int_\Lambda f(\lambda)\xi(\mathrm{d}\lambda)$, which is defined precisely in Appendix A.

(*i*) $\forall p \in \mathbb{N}_{0+}$, $\alpha_{2p} := (2p)!/2^p \cdot p!$. Thus

$$\alpha_0 = 1 \quad \& \quad \forall p \in \mathbb{N}, \qquad \alpha_{2p} = (2p-1)\cdot(2p-3)\cdots 3\cdot 1. \tag{1.2}$$

The α_{2p} increase very rapidly: $\alpha_0 = \alpha_2 = 1$, $\alpha_4 = 3$, $\alpha_6 = 15$, $\alpha_8 = 105$, $\alpha_{10} = 945$, ...

(*d*) *Wiener's p-homogeneous chaotic measure*

We must first comment on the simpler and more basic random variable-valued measure ξ over $\mathbb{R}$ given by (2), the p-fold product of which constitutes the measure ξ_p under investigation. To adopt a more general setting, let

$$\begin{cases} \Lambda \text{ be a locally compact additive abelian (l.c.a.) group;} \\ \mathcal{D} \text{ be the } \delta\text{-ring of a Borel subsets } D \subseteq \Lambda \text{ with compact closures;} \\ \mathcal{D}^{\mathrm{loc}} := \{A : A \subseteq \Lambda \ \& \ \forall D \in \mathcal{D}, A \cap D \in \mathcal{D}\}; \\ \ell \text{ be the Haar measure on } \mathcal{D}\text{; thus } \ell \in \mathrm{CA}(\mathcal{D}, \mathbb{R}_{0+}); \\ (\Omega, \mathcal{A}, \mathbb{P}) \text{ be a probability space and } \mathcal{L}_2 := L_2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}). \end{cases} \tag{1.3}$$

As Kakutani (1961, p. 241, ex. 2), has observed, one can define a Λ-analogue on $\mathcal{D}$

of the extension of the *random measure* over $\mathbb{R}$ defined by (2) in terms of Wiener's Brownian motion SP, as follows:

1.4. *Definition.* Let (i) Λ, $\mathcal{D}$, ℓ, $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{L}_2$ be as in (1.3). We say that ρ is the *Brownian motion random measure* (BMRM) on $\mathcal{D}$ *with probability space* $(\Omega, \mathcal{A}, \mathbb{P})$, if and only if

(*a*) $\forall D \in \mathcal{D}$, $\rho(D)$ is a $\mathbb{R}$-valued, normally distributed random variable on $(\Omega, \mathcal{A}, \mathbb{P})$ with mean 0 and variance $\ell(D)$;

(*b*) ρ is finitely additive on $\mathcal{D}$;

(*c*) ρ is independently scattered, i.e.

$$\forall n \in \mathbb{N}_+ \quad \text{and} \quad \forall \| D_1, \ldots, D_n \in \mathcal{D}, \quad \rho(D_1), \ldots, \rho(D_n)$$

are stochastically independent.

The existence of this BMRM is easy to show once the concepts of a *Gaussian system* and *Gaussian subspace*, due to Kakutani (1961), are demarcated:

1.5. *Definition.* Let $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{L}_2$ be as in (1.3). Then

(*a*) $\mathcal{G}$ is called a *Gaussian system*, if and only if $\mathcal{G} \subset \mathcal{L}_2$, and every finite linear combination of vectors in $\mathcal{G}$ is normally distributed;

(*b*) $\mathcal{G}$ is called a *Gaussian subspace* of $\mathcal{L}_2$, if and only if (i) $\mathcal{G}$ is a closed linear subspace of $\mathcal{L}_2$, and (ii) each random variable $x(\cdot)$ in $\mathcal{G}$ is normally distributed.

Since Gaussian systems are independent if and only if they are uncorrelated, the notions of 'independently scattered' (IS) and 'orthogonally scattered' (OS) measures coincide for zero means; and using the known result on equivalence of OS measures (cf. Masani 1968, 1.8), and the disjoint normal form for sets, we get the following lemma on equivalence:

1.6. Lemma. *With the notation of (1.3), ρ is the BMRM on $\mathcal{D}$ with probability space* $(\Omega, \mathcal{A}, \mathbb{P})$*, in the sense of 1.4, if and only if* (i) $\{\rho(D) : D \in \mathcal{D}\}$ *is a Gaussian system and each* $\xi(D)$ *has mean 0, and* (ii) $\rho \in \mathrm{CAOS}(\mathcal{D}, \mathcal{L}_2)$*, i.e.*

$$\rho \in \mathrm{CA}(\mathcal{D}, \mathcal{L}_2) \quad \& \quad \forall D, E \in \mathcal{D}, \qquad (\rho(D), \rho(E))_{\mathcal{L}_2} = \ell(D \cap E).$$

It follows from a theorem of Doob (1953, p. 72), that

$$(1.7) \quad \begin{cases} \forall \text{ cardinal numbers } \alpha, \exists \text{ a probability space } (\Omega, \mathcal{A}, \mathbb{P}) \ \& \ \exists \text{ a Gaussian} \\ \text{subspace } \mathcal{G} \text{ of } \mathcal{L}_2 \text{ such that } \dim \mathcal{G} = \alpha \ \& \ \forall y \in \mathcal{G},\ \mathbb{E}_{\mathbb{P}}(y) = 0. \end{cases}$$

1.8. Proposition. (Existence of BMRM) *Let Λ, $\mathcal{D}$, ℓ be as in (1.3). Then $\exists$ a probability space* $(\Omega, \mathcal{A}, \mathbb{P})$ *and $\exists$ a BMRM ρ on $\mathcal{D}$ with probability space* $(\Omega, \mathcal{A}, \mathbb{P})$.

Proof. Following Kahane (1968, pp. 154–155), let $\mathcal{H} := L_2(\Lambda, \mathcal{D}, \ell; \mathbb{R})$ and $\alpha := \dim \mathcal{H}$. By (1.7), $\exists(\Omega, \mathcal{A}, \mathbb{P})$ and $\exists$ a Gaussian subspace $\mathcal{G} \subseteq \mathcal{L}_2 = L_2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R})$ such that $\dim \mathcal{G} = \alpha$ and $\forall y \in \mathcal{G}$, $\mathbb{E}_{\mathbb{P}}(y) = 0$. Let V be any linear isometry on $\mathcal{H}$ onto $\mathcal{G}$, and define $\forall D \in \mathcal{D}$, $\rho(D) := V(\chi_D)$. Then obviously $\{\rho(D) : D \in \mathcal{D}\}$ is a Gaussian system, $\rho(\cdot)$ is CA on $\mathcal{D}$, and

$$\forall D, E \in \mathcal{D}, \qquad (\rho(D), \rho(E))_{\mathcal{L}_2} = (\chi_D, \chi_E)_{\mathcal{H}} = \ell(D \cap E).$$

Finally, $\mathbb{E}_{\mathbb{P}}\{\rho(D)\} = 0$. Hence, by lemma 1.6, ρ is the desired BMRM. ∎

The fact that the BMRM $\rho(\cdot)$ is CAOS makes the theory of the integration $\mathbb{E}_\rho$

especially simple and elegant. The fact that the control measure ℓ of ρ is Haar, makes (cf. Masani 1968, § 7) ρ a *stationary measure* in the sense of Masani (1983, def. 2.3(c)).

The hierarchy of measures ξ_p, $p \in \mathbb{N}_+$, that is latent in Wiener's paper [W], can be developed by starting from a BMRM ξ over any l.c.a. group Λ. The resulting theory is in essence the same as the one based on $\Lambda = \mathbb{R}$, that Wiener considered in [W] and which we shall follow in the paper. The following notation is germane to this $\mathbb{R}$-based theory:

1.9. *Notation.* (Set families over $\mathbb{R}^p$, for $p \in \mathbb{N}_+$.)

$$\begin{aligned}
\mathcal{D}_p &:= \text{the } \delta\text{-ring of bounded Borel subsets of } \mathbb{R}^p, \\
\mathcal{B}_p &:= \text{the } \sigma\text{-ring of Borel subsets of } \mathbb{R}^p \text{ (a } \sigma\text{-algebra)}, \\
\mathcal{P}_p &:= \left\{ \bigtimes_{i=1}^{p} P^i : P^i \in \mathcal{D}_1 \right\} \text{ is the pre-ring of } \mathcal{D}_1\text{-edged intervals}, \\
\forall \mathcal{F} \subseteq 2^{\mathbb{R}^p}, &\quad \mathcal{F}^{\text{sym}} = \{F : F \in \mathcal{F} \text{ and } F \text{ is symmetric}\} \\
&\quad \text{(thus we shall have } \mathcal{D}_p^{\text{sym}}, \mathcal{B}_p^{\text{sym}}, \mathcal{P}_p^{\text{sym}}), \\
\ell_p &:= \text{the Lebesgue measure on } \mathcal{D}_p, \\
\bar{\mathcal{D}}_p &:= \{B : B \in \mathcal{D}_p^{\text{loc}} \ \& \ |\ell_p|(B) < \infty\}.^{1}
\end{aligned}$$

Whereas $\mathcal{P}_1 = \mathcal{D}_1$ is a δ-ring over $\mathbb{R}$, for $p \geqslant 2$, $\mathcal{P}_p$ is only a pre-ring, namely, the pre-ring of intervals P in $\mathbb{R}^p$, the edges P^i of which are bounded Borel subsets of $\mathbb{R}$. Obviously,

$$\mathcal{P}_p \subseteq \mathcal{D}_p \subseteq \bar{\mathcal{D}}_p = \text{a } \delta\text{-ring} \subseteq \mathcal{B}_p = (\mathcal{D}_p)^{\text{loc}}.$$

Let

(1.10) $$\forall p \in \mathbb{N}_+, \qquad \mathcal{R}_p := \text{ring}(\mathcal{P}_p) = \text{the ring generated by } \mathcal{P}_p.$$

It is well known (see, for example, [MN, I, triv. A.4(a)]) that

(1.11) $$\forall p \in \mathbb{N}_+, \qquad \delta\text{-ring}(\mathcal{P}_p) = \delta\text{-ring}(\mathcal{R}_p) = \mathcal{D}_p.$$

Now let ξ be the *Brownian motion random measure* over $\mathbb{R}$, cf. 1.4. Then by lemma 1.8,

(1.12) $$\begin{cases} \forall D \in \mathcal{D}_1, \quad \xi(D) \in \mathcal{L}_2 \text{ is normally distributed} \\ \qquad\qquad\qquad\qquad \text{with mean 0 and variance } \ell_1(D), \\ \xi \text{ is CA on } \mathcal{D}_1 \ \& \ \forall D, E \in \mathcal{D}_1, \quad (\xi(D), \xi(E)) = \ell_1(D \cap E). \end{cases}$$

Briefly,

$$\xi \in \text{CAOS}(\mathcal{D}_1, \mathcal{L}_2) \quad \& \quad \mathbb{E}_{\mathbb{P}}\{\xi(\cdot)\} = 0 \quad \text{on} \quad \mathcal{D}_1.$$

The pth measure underlying Wiener's theory is defined to be the ξ_p whose value at the p-dimensional interval $P := P^1 \times \cdots \times P^p$, where $P^k \in \mathcal{D}_1$, is the random variable

[4] The natural domain of a vector measure ρ is a δ-ring $\mathcal{D}$, but the natural domains of the quasi-, semi- and total-variations are the σ-algebra $\mathcal{D}^{\text{loc}}$ of (1.3). It is convenient to maintain this distinction between measure and total variation even for non-negative real-valued measures such as ℓ_p. Thus $|\ell_p|$ is defined on $\mathcal{B}_p$ and for $B \in \mathcal{B}_p$, $|\ell_p|(B) = \sup_{D \in \mathcal{D}_p} \ell_p(B \cap D) \in [0, \infty]$. Obviously, $\ell_p \subseteq |\ell_p| \in \text{CA}(\mathcal{B}_p, [0, \infty])$.

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having at each $\omega \in \Omega$ the value

$$(1.13) \qquad \xi_p(P)(\omega) := \prod_{k=1}^{p} \{\xi(P^k)\}(\omega).$$

Since normally distributed random variables have moments of all orders, it is easy to see that $\xi_p(P) \in \mathcal{L}_2$, and that, in fact,

$$(1.14) \qquad \xi_p \in \mathrm{FA}(\mathcal{P}_p, \mathcal{L}_2).$$

In our approach the key to the understanding of the FA measures ξ_p lies in first finding their cross-covariance, i.e. in answering the following question:

1.15. *Question.* Let $p, q \in \mathbb{N}_+$, $D \in \mathcal{D}_p$ & $E \in \mathcal{D}_q$. Then how is the RHS of the cross-covariance formula

$$(\xi_p(D), \xi_q(E))_{\mathcal{L}_2} = \cdots\cdots$$

to be completed in terms of familiar scalar measures?

This question has to be answered first for intervals D, E in $\mathcal{P}_p$, $\mathcal{P}_q$, respectively, and then for sets in the rings $\mathcal{R}_p$, $\mathcal{R}_q$. This done, and the standard conditions for countable-additivity and extendibility having been shown, we must answer the question for arbitrary D and E in $\mathcal{D}_p$ and $\mathcal{D}_q$, respectively.

(*e*) *On Wiener's equality [W, (75)]*

The key to answering Question 1.15 lies in the systematic and extended utilization of the equality (75) which Wiener gave in [W] for the expectation of the product $\xi(A_1) \cdot \xi(A_2) \cdots \xi(A_{2n})$, where ξ is the BMRM over $\mathbb{R}$, and $A_1, \ldots, A_{2n} \in \mathcal{D}_1$. In reading Wiener's words, note that $\mathcal{P}$ is his symbol for the BMRM over $\mathbb{R}$, and Σ_i are his sets in $\mathcal{D}_1$ and his $(\Omega, \mathcal{B}, \mathbb{P})$ is [0,1] with Lebesgue measure; thus his $\mathcal{P}(\Sigma_i, \alpha)$ is our $\xi(A_i)(\omega)$. Wiener wrote:

> Remembering that if $S_1, S_2, \ldots, S_{2n}$ are non-overlapping, their distributions are independent, we see that if the sets $\Sigma_1, \Sigma_2, \ldots, \Sigma_{2n}$ are either totally non-overlapping, or else such that when two overlap, they coincide, we have
>
> $$(75) \qquad \int_0^1 \mathcal{P}(\Sigma_1; \alpha) \cdots \mathcal{P}(\Sigma_{2n}; \alpha)\,\mathrm{d}\alpha = \sum \prod \int_0^1 \mathcal{P}(\Sigma_j; \alpha)\mathcal{P}(\Sigma_k; \alpha)\,\mathrm{d}\alpha,$$
>
> where the product sign indicates that the $2n$ terms are divided into n sets of pairs, j and k, and that these factors are multiplied together, while the addition is over all the partitions of $1, \ldots, 2n$ into pairs. If $2n$ is replaced by $2n+1$, the integral in (75) of course vanishes.
>
> Since $\mathcal{P}(\mathcal{S}; \alpha)$ is a linear functional of sets of points, and since both sides of (75) are linear with respect to each $\mathcal{P}(\Sigma_k; \alpha)$ separately, (75) still holds when $\Sigma_1, \Sigma_2, \ldots, \Sigma_{2n}$ can be reduced to sums of sets which either coincide or do not overlap, and hence *holds for all measurable sets.*
>
> ([W, p. 917]; emphasis added)[5]

In this the crucial idea is that of 'partitioning $2n$ terms into pairs'. It is a remarkably deep and resilient idea, which works even for a body E in $\mathbb{R}^p$, such as an ellipsoid,

[5] Wiener's use of the term 'functional' becomes intelligible on noting that it was his habit to switch (without warning) from $\mathcal{P}(\Sigma_i; \alpha)$ to the equivalent formulation $\{\mathbb{E}_{\mathcal{P}}(\chi_{\Sigma_i})\}(\alpha)$.

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once meaningful entities E_{ij} canonically associated with E and the pairs (i,j) are demarcated. Its systematic analysis requires the following notation:

1.16. *Notation.* (Binary-celled partitions)

(a) $\forall k\in\mathbb{N}_+$, and all non-void sets M of even cardinality $2k$,

$$\Pi_M := \{\pi : \pi \text{ is a partition of } M \text{ into binary cells}\};\qquad \Pi_\emptyset := \{\emptyset\};$$

(b) the cells Δ of π in Π_M will be so numbered:

$$\pi = \{\Delta_1,\Delta_2,\ldots,\Delta_k\},\quad \text{that}\quad \forall\alpha\in[1,k-1],\quad \min\Delta_\alpha \leqslant \min\Delta_{\alpha+1};$$

(c) $\forall\pi\in\Pi_M$, ${}^*\pi := \{\min\Delta : \Delta\in\pi\}$, $\pi^* := \{\max\Delta : \Delta\in\pi\}$;

$${}^*\emptyset := \emptyset =: \emptyset^*;$$

(d) $\forall p\in\mathbb{N}_+\quad \&\quad \forall k\in[1,[p/2]]$,

$$\Pi_k^p := \bigcup_{\substack{M\subseteq[1,p]\\ \#M=2k}} \Pi_M;$$

thus $\forall k\in[0,[p/2]]$, Π_k^p is the class of all binary-celled partitions of all subsets of $[1,p]$ of cardinality $2k$;

(e) $\forall p\in\mathbb{N}_+$, $\forall k\in[1,[p/2]]$ and $\forall\pi\in\Pi_k^p$,

$$M_\pi := \bigcup_{\Delta\in\pi}\Delta,\qquad M'_\pi := [1,p]\backslash M_\pi.$$

Simple combinatory considerations show that with the notation 1.1(i),

$$\text{(1.17)}\qquad \begin{cases} \forall p\in\mathbb{N}_+, \forall k\in[0,[p/2]]\quad \&\quad \forall\pi\in\Pi_k^p,\quad \#\pi = k,\\ \forall q\in\mathbb{N}_+,\quad \Pi_k^p\subseteq\Pi_k^{p+q}\quad \&\quad \forall \text{ even } p\in\mathbb{N}_+,\quad \Pi_{p/2}^p = \Pi_{[1,p]},\\ \forall k\in\mathbb{N}_{0+},\quad \#M = 2k \Longrightarrow \#\Pi_M = \alpha_{2k};\quad \#\Pi_k^p = \dbinom{p}{2k}\alpha_{2k},\\ \forall p\in\mathbb{N}_+,\quad \Pi_0^p = \{\emptyset\};\quad \text{in particular } \Pi_0^1 = \{\emptyset\}. \end{cases}$$

With this notation Wiener's equality (75) can be rendered as follows:

1.18. Theorem. (Wiener's equality, form 1) *Let $n\in\mathbb{N}_+$ and $A_1,\ldots,A_n\in\mathcal{D}_1$. Then, cf.* 1.1($h$),

(a) *for n odd,*

$$\mathbb{E}_{\mathbb{P}}\left\{\prod_{i=1}^{n}\xi(A_i)\right\} = 0;$$

(b) *for n even $=2r$,*

$$\mathbb{E}_{\mathbb{P}}\left\{\prod_{i=1}^{2r}\xi(A_i)\right\} = \sum_{\pi\in\Pi_{[1,2r]}}\prod_{\Delta\in\pi}(\xi(A_{\min\Delta}),\xi(A_{\max\Delta}))_{\mathcal{L}_2}.$$

Proof. A proof along the lines Wiener indicated in [W], is given in § 2. ∎

In terms of the measures ξ_p, theorem 1.18 reduces to:

1.19. Corollary. (The expectation of ξ_p on $\mathcal{P}_p$) *Let $p\in\mathbb{N}_+$ and $P := P^1\times\cdots\times P^p\in\mathcal{P}_p$. Then*

(*a*) *for odd p,*
$$\mathbb{E}_{\mathbb{P}}\{\xi_p(P)\} = 0;$$

(*b*) *for even p,*
$$\mathbb{E}_{\mathbb{P}}\{\xi_p(P)\} = \sum_{\pi \in \Pi_{[1,p]}} \prod_{\Delta \in \pi} (\xi(P^{\min \Delta}), \xi(P^{\max \Delta}))_{\mathcal{L}_2};$$

(*c*) *for even p,*
$$\mathbb{E}_{\mathbb{P}}\{\xi_p(P)\} = \sum_{\pi \in \Pi_{[1,p]}} \prod_{\Delta \in \pi} \ell_1\{P(\Delta)\} \in \mathbb{R}_{0+}, \quad P(\Delta) := P^{\min \Delta} \cap P^{\max \Delta};$$

cf. (1.12);

(*d*) *for even p and* $A \in \mathcal{D}$, $\mathbb{E}_{\mathbb{P}}\{\xi_p(A^P)\} = \alpha_{p/2}\ell_{p/2}(A^{p/2})$; *cf. [W, eqn (74)].*

The RHS of the equalities 1.19(*b*), (*c*) involve the p edges $P^1, \ldots, P^p$ of P. It is not clear what the RHS can possibly mean when P is not an interval, but some other body in $\mathbb{R}^p$, such as an ellipsoid. We shall show that there is a suitable re-rendering of the product on the RHS of 1.19(*b*), however, whereby it will continue to make sense for any $D \in \mathcal{D}_p$. This requires the intersection of D with the first 'diagonal skeleton':

$$I_1^p := \bigcup_{i=1}^{p} \bigcup_{j=i+1}^{p} I_{i,j}, \qquad I_{i,j} := \{x : x \in \mathbb{R}^p \ \&\ x_i = x_j\}, \tag{1.20}$$

with which the space $\mathbb{R}^p$ is naturally endowed. Wiener's equality (75) survives remarkably well.

This quite essential intrusion of the first and also higher order diagonal skeletons, cf. (4.5), brings into focus novel aspects of the anatomy of the spaces $\mathbb{R}^p$ not revealed by the study of their coordinate hyperplanes of dimensions $p-1$, $p-2, \ldots, 2, 1$. It interjects into the theory a large amount of complicated combinatorics, which demands the introduction of several combinatory concepts. Thankfully, the extremely technical combinatory analysis usually ensues in some aesthetically satisfactory result.

(*f*) *New results*

In § 3 we first show that for $p \geqslant 2$, the measure ξ_p deviates from ξ_1, in not being absolutely continuous with respect to ℓ_p (cf. 3.8). We then answer the basic question 1.15 for intervals P, Q in 3.13. In § 4 we bring in the diagonal skeleton I_i^p and give a reinterpretation in terms of it for the RHS of the cross-covariance formula in 3.13, whereby it continues to make sense for any $D \in \mathcal{D}_p$ and any $E \in \mathcal{D}_q$. This is done by defining for each $D \in \mathcal{D}_p$ and each $k \in [1, [p/2]]$ and each $h \in \mathbb{R}^{p-2k}$, *canonical coefficients* $\gamma_k^p(D, h)$, cf. definition 4.13, and showing that the RHS of the cross-covariance formulae in (3.13) with $q \leqslant p$ is a sum of integrals of the type,

$$\int_{\mathbb{R}^{q-2k}} \gamma^p_{\frac{1}{2}(p-q)+k}(D,h)\gamma_k^q(E,h^{\phi})\ell_{q-2k}(\mathrm{d}h),$$

h^{ϕ} being a permutation of the components of h.

In § 5 we show that the reframed cross-covariance equality holds for $D \in \mathcal{R}_p$ and $Q \in \mathcal{R}_q$. Immediate corollaries are the countable additivity and strong additivity of ξ_p on $\mathcal{R}_p$ and its (countably additive) extendibility to $\mathcal{D}_p$. A little extra effort shows the validity of the covariance formula itself for $D \in \mathcal{D}_p$ and $E \in \mathcal{D}_q$. This accomplishes the first major objective of the paper. The equality $\xi_p \times \xi_q = \xi_{p+q}$ on $\mathcal{D}_p \times \mathcal{D}_q$ follows.

Our next major objective, quite difficult, is to show that the (closed) subspaces $\mathcal{S}_{\xi_p}$ of $\mathcal{L}_2$ spanned by the values of the measures ξ_p on $\mathcal{D}_p$ for $p \geqslant 0$, satisfy the strict inclusions,

$$(1.21)\qquad \begin{cases} \mathcal{S}_{\xi_0} \subset \mathcal{S}_{\xi_2} \subset \cdots \subset \mathcal{S}_{\xi_{2p}} \subset \cdots; \quad \mathcal{S}_{\xi_1} \subset \mathcal{S}_{\xi_3} \subset \cdots \subset \mathcal{S}_{\xi_{2p+1}} \subset \cdots; \\ \qquad \mathcal{S}_{\xi_{2p}} \perp \mathcal{S}_{\xi_{2q+1}}, \quad p, q \in \mathbb{N}_{0+}. \end{cases}$$

Here ξ_0 is the measure on $\{\emptyset, \{0\}\}$, where $0 \in \mathbb{R}$, such that $\xi_0(\emptyset) = 0$ and $\xi_0\{(0)\}$ is the function constantly 1 on Ω (§§ 7, 8). With this new information, we show that the orthogonal projections η_p, ζ_p of the measures ξ_p on $\mathcal{S}^{\perp}_{\xi_{p-2}}$ and $\mathcal{S}_{\xi_{p-2}}$, respectively, are precisely the absolutely continuous and singular parts of the measure ξ_p with respect to the Lebesgue measure ℓ_p. These projections thus yield the Lebesgue decomposition of ξ_p with respect to ℓ_p.

The covariance structure of the measure η_p, the absolutely continuous part of ξ_p, is considerably simpler than that of ξ_p; we have

$$\mathcal{S}_{\eta_p} \perp \mathcal{S}_{\eta_q}, \quad p \neq q; \quad \& \quad \forall D, E \in \mathcal{D}_p, \quad (\eta_p(D), \eta_p(E))_{\mathcal{L}_2} = \sum_{\phi} \ell_p(D \cap E^{\phi}),$$

where the summation is over the class of all permutations ϕ of $\{1, 2, \ldots, p\}$ and E^{ϕ} is the ϕ-permutation of the set E. Letting $\mathcal{D}_p^{\mathrm{sym}}$ be as in (1.9), it follows at once that

$$\forall D, E \in \mathcal{D}_p^{\mathrm{sym}}, \quad (\eta_p(D), \eta_p(E))_{\mathcal{L}_2} = p!\,\ell_p(D \cap E),$$

i.e. η_p, like ξ_1, cf. (1.12), is *orthogonally scattered* on $\mathcal{D}_p^{\mathrm{sym}}$. Connecting ξ_p and η_p, we have the orthogonal decomposition,

$$\mathcal{S}_{\xi_p} = \mathcal{S}_{\eta_p} + \mathcal{S}_{\eta_{p-2}} + \cdots + \mathcal{S}_{\eta_{p-2[p/2]}}, \quad \mathcal{S}_{\eta_j} \perp \mathcal{S}_{\eta_k}, \quad j \neq k,$$

the final term being $\mathcal{S}_{\eta_1}$ or $\mathcal{S}_{\eta_0}$ according as p is odd or even (§ 9).

Turning to the Lebesgue–Pettis integrability and integration, we first deal with the class $\mathcal{P}_{1,\eta_p}$ of all real-valued Lebesgue–Pettis integrable functions with respect to the simpler measure η_p, and the Lebesgue–Pettis integral operator $\mathbb{E}_{\eta_p}$ (§ 10). We prove that

$$\mathcal{P}_{1,\eta_p} = L_2(\mathbb{R}^p).$$

In this the inclusion $\mathcal{P}_{1,\eta_p} \subseteq L_2(\mathbb{R}^p)$ is far from obvious. We then show that

$$\frac{1}{\sqrt{p!}}\mathbb{E}_{\eta_p} = \text{a partial isometry on } L_2(\mathbb{R}^p) \text{ onto } \mathcal{S}_{\eta_p} \subseteq \mathcal{L}_2,$$

the null space of which is the class of all functions in $L_2(\mathbb{R}^p)$ with vanishing symmetrization, and that therefore the restriction of $(1/\sqrt{p!})\mathbb{E}_{\eta_p}$ to the class $L_2^{\mathrm{sym}}(\mathbb{R}^p)$ of symmetric functions in $L_2(\mathbb{R}^p)$ is an isometry on $L_2^{\mathrm{sym}}(\mathbb{R}^p)$ onto $\mathcal{S}_{\eta_p} \subseteq \mathcal{L}_2$. Letting

$$\mathcal{L}_2^{\xi} := \operatorname{cls} \bigcup_{k=0}^{\infty} \mathcal{S}_{\eta_k},$$

we obtain the following explicit orthogonal expansion:

$$(1.22)\qquad \forall x \in \mathcal{L}_2, \quad \operatorname{proj}(x|\mathcal{L}_2^{\xi}) = \sum_{p=0}^{\infty} \frac{1}{p!}\mathbb{E}_{\eta_p}(f_x^p),$$

where $f_x^p := \mathrm{d}\nu_x^p/\mathrm{d}\ell_p$ stands for the Radon–Nikodym derivative, and $\nu_x^p(\Delta) := (x, \eta_p(\Delta))$, $\Delta \in \mathcal{D}_p$ (§ 11).

Integration with respect to the vector measure η_p yields a full-fledged theory of projections of $\xi_p(D)$ onto $\mathcal{S}_{\xi_q}$: we show that for $p, q \in \mathbb{N}_+$, such that $q \leqslant p$ and $p-q$ is even,

$$\mathrm{Proj}(\xi_p(D)|\mathcal{S}_{\xi_q}) = \sum_{k=0}^{[q/2]} \int_{\mathbb{R}^{q-2k}} \gamma^p_{\frac{1}{2}(p-q)+k}(D,h)\eta_{q-2k}(\mathrm{d}h), \quad D \in \mathcal{D}_p.$$

In particular, we have for $q = p$,

$$\xi_p(D) = \sum_{k=0}^{[p/2]} \int_{\mathbb{R}^{p-2k}} \gamma^p_k(D,h)\eta_{p-2k}(\mathrm{d}h). \tag{1.23}$$

Turning to the more involved integrability class $\mathcal{P}_{1,\xi_p}$ and the more involved integration $\mathbb{E}_{\xi_p}$, we first so define the p, k marginalization f^p_k of f that

$$f^p_k(\cdot) := \int_{\mathbb{R}^p} f(t)\gamma^p_k(\mathrm{d}t,\cdot), \quad \text{a.e.} \quad \ell_{p-2k} \quad \text{on} \quad \mathbb{R}^{p-2k} \qquad (\S\,12) \tag{1.24}$$

and show that $f \in \mathcal{P}_{1,\xi_p}$ iff for each k, $f \in L_{1,\gamma^p_k(\cdot,h)}$ and $f^p_k(\cdot) \in \mathcal{P}_{1,\eta_{p-2k}}$, and that for $f \in \mathcal{P}_{1,\xi_p}$,

$$\mathbb{E}_{\xi_p}(f) = \sum_{k=0}^{[p/2]} \mathbb{E}_{\eta_{p-2k}}(f^p_k) \qquad (\S\,13). \tag{1.25}$$

From the formula (1.25) the cross-covariance $(\mathbb{E}_{\xi_p}(f), \mathbb{E}_{\xi_q}(g))$, for $f \in \mathcal{P}_{1,\xi_p}$, $g \in \mathcal{P}_{1,\xi_q}$, is easily obtained as is also the formula for the expectation $\mathbb{E}_{\mathbb{P}}\{\mathbb{E}_{\xi_p}(f)\}$.

The next major objective, the Fubini theorem for the tensor product, hinges on the implication

$$f \in \mathcal{P}_{1,\xi_p} \quad \& \quad g \in \mathcal{P}_{1,\xi_q} \Longrightarrow f \times g \in \mathcal{P}_{1,\xi_{p+q}}, \quad p, q \in \mathbb{N}_+. \tag{1.26}$$

But this is extremely hard to show, since nothing is known about the action of a linear functional on a product of vectors (see 14.10 and infra). Once (1.26) is established, it follows easily that $\mathbb{E}_{\xi_{p+q}}(f \times g) = \mathbb{E}_{\xi_p}(f) \cdot \mathbb{E}_{\xi_q}(g)$.

The inversion of the relations (1.23) and (1.25) turn out to be, respectively

$$\eta_p(D) = \sum_{k=0}^{[p/2]} (-1)^k \int_{\mathbb{R}^{p-2k}} \gamma^p_k(D,h)\xi_{p-2k}(\mathrm{d}h) \tag{1.27}$$

and

$$\mathbb{E}_{\eta_p}(f) = \sum_{k=0}^{[p/2]} (-1)^k \int_{\mathbb{R}^{p-2k}} f^p_k(h)\xi_{p-2k}(\mathrm{d}h). \tag{1.28}$$

These Möbius inversions are very difficult to prove (§ 15), and require a new 'division' operation $A|B$ for finite sets such that $A \subseteq B$, discussed in Appendix C. These inversions establish a nexus between the η_p and $\mathbb{E}_{\eta_p}$ and the Hermite polynomials in the Kakutani format (16.2), namely,

$$\forall A \in \mathcal{D}_1, \qquad \eta_p(A^p) = H_p\{\xi_1(A), \ell_1(A)\}$$

and

$$\forall f \in L_2(\mathbb{R}) \quad \& \quad f^{\times p} = f \times f \times \cdots \times f \quad (p \text{ times}), \quad \mathbb{E}_{\eta_p}(f^{\times p}) = H_p\{\mathbb{E}_{\xi_1}(f), |f|^2_{2,\ell_1}\},$$

where the exponent $\times p_i$ indicates tensor power (§ 16). Combining this with the Fubini equality, we have for mutually orthogonal $f_1, \ldots, f_n \in L_2(\mathbb{R})$,

$$\mathbb{E}_{\eta_{p_1+\cdots+p_n}}\left(\bigtimes_{i=1}^{n} f_i^{\times p_i}\right) = \prod_{i=1}^{n} H_{p_i}\{\mathbb{E}_{\xi_1}(f_i), |f_i|^2_{2,\ell_1}\}, \tag{1.29}$$

The last equality allows us to deduce the results of Ito, Kakutani, and Cameron & Martin (§ 16).

(*g*) *Limitations of the theory and further work*

The theory given in this paper fails in regard to the general Fubini theorem. Even in the simplest case, $p = q = 1$, this theorem, to wit, $\forall F \in \mathcal{P}_{1,\xi_2}$,

$$\mathbb{E}_{\xi_2}(F) = \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} F(s,t)\xi_1(\mathrm{d}s) \right\} \xi_1(\mathrm{d}t) \tag{5}$$

cannot be articulated, since the integrand in (5), namely, the partial integral $G(\cdot)$ defined on $\mathbb{R}$ by

$$\forall t \in \mathbb{R}, \qquad G(t) = \int_{\mathbb{R}} F(s,t)\xi_1(\mathrm{d}s) \in \mathcal{L}_2$$

is not scalar-valued but random-variable-valued, and its integration falls outside the ambit of Appendix A. The same difficulty afflicts the general slicing equality

$$\xi_{p+q}(D) = \int_{\mathbb{R}^q} \xi_p(D^t)\xi_q(\mathrm{d}t).$$

To overcome this limitation the integration pattern outlined in Appendix A will have to be developed for the case in which both measure and integrand are random variables possessing finite moments of all orders. It would be interesting to know how the 'stochastic integration', so resulting, will connect with the important and widely used stochastic integration initiated by Ito (1944).

Also opened up for future investigation are the (univariate) distribution functions $F_{p,D}(\cdot)$ on $\mathbb{R}$ of the $\mathbb{R}$-valued random variable $\xi_p(D)$, for $p \in \mathbb{N}_+$ and $D \in \mathcal{D}_p$. What is $F_{p,D}(\cdot)$, for instance, when $p = 3$ and D is the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$? Does $F_{3,D}$ have a density? Is it determined by its moments? Is it infinitely decomposable?

An extension in a different direction would be to work out the theory starting with complex-valued $\rho(D)$ in definition 1.4(a). It might shed new light on the quantum mechanical bearing of the Brownian motion (cf. Wiener 1985, lec. 9; Segal 1956).

(*h*) *Bearing on the Feynman integral*

The equalities (1.24), (1.25), which depart from the existing vector measure theory, bear significantly on the recent efforts of Hu & Meyer (1980) to explicate mathematically the Feynman integral. In their paper (1980, eqn (5)), and in the later paper on this subject by Johnson & Kallianpur (1993) appears the so-called kth trace of a function f on $\mathbb{R}^p$, which is defined on $\mathbb{R}^{p-2k}$ by

$$(\mathrm{Tr}^k f)(s_{2k+1}, \ldots, s_p) = \int_{\mathbb{R}^p} f(s_1, s_1, \ldots, s_k, s_k; s_{2k+1}, \ldots, s_p)\, \mathrm{d}s_1 \cdots \mathrm{d}s_k. \tag{6}$$

The p, k marginalization f_k^p, given in (1.24), which appears quite naturally in the theory of $\mathbb{E}_{\xi_p}$, is exactly $\binom{p}{2k}\alpha_{2k}$ times this k-trace, in the special case in which f is symmetric, cf. 12.18b, e below. Its existence is ensured by the condition $f \in \mathcal{P}_{1,\xi_p}$, i.e. by the requirement that f be Lebesgue–Pettis integrable with respect to the (full fledged) chaotic measure ξ_p. The imposition of this condition automatically provides explicit liftings, cf. 13.18 and 13.19, and obviates the need for *ad hoc* searches of the kind undertaken in recent papers on the Feynman integral.

(*i*) *Historical remarks*

While Wiener [W] did not prove the countable additivity of ξ_p, he left enough of a clue in the formula (75) whereby a modern researcher could get to it by first addressing the covariance question 1.15. As for the so-called 'multiple Wiener integral', Wiener's attempt to introduce it ([W], eqs (76)–(87)) is flawed.[6] His treatment in both [W] and his book (1958) is incomplete by virtue of his silence on the integrability classes $\mathcal{P}_{1,\xi_p}$, and his unawareness that the implication (1.26) needed demonstration. But after these lacunae are filled in, his conclusions (including the one derived from the flawed equations in [W]) are seen to be completely correct. See corollary 14.12 below.

Integrals, in which pairs of variables in the integrand are identical, needed for the Feynman work, play an intrinsic role in Wiener's 1938 paper [W] and more so in his 1958 book. Thus equation (77) in [W], where n is even and $n = 2m$, reads (after a typographical correction),

$$\mathbb{E}_{\mathbb{P}}\left\{\int_{\mathbb{R}^n} f(t_1,\ldots,t_n)\xi(\mathrm{d}t_1)\cdots\xi(\mathrm{d}t_n)\right\} = \sum \int_{\mathbb{R}^m} f(t_1,t_1,\ldots,t_m,t_m)\,\mathrm{d}t_1\cdots\mathrm{d}t_n,$$

'where the summation is carried out for all possible division of the $2m$ ts into pairs', and where it should be added the integrand exhibited on the RHS corresponds to the simplest of all such pairings. Such summations also appear in Wiener's book *Cybernetics* (1961, eqns (3.22), (3.23)). Situations such as those in the trace formula (6), p. 1111, where the pairing is only of $2k$ out of n variables, are implicit in [W] and appear explicitly in the 1958 book, cf., for example, equations (3.20), (3.31) for the case $n = 2, 3$ and the remarks for larger n on page 35 (bottom).

Furthermore, Wiener derives the integrations $\mathbb{E}_{\eta_2}$, $\mathbb{E}_{\eta_3}$ (his G_1, G_2) from $\mathbb{E}_{\xi_2}$, $\mathbb{E}_{\xi_3}$ by a Gram-Schmidt procedure in Wiener (1958, pp. 28–36), getting in effect the Möbius inversions (1.28). But Wiener worked only with the very easy cases $p = 2$ and $p = 3$, and rather cavalierly dispensed with the troubles latent in handling arbitrary p by a remark or two (cf., for example, Wiener 1958, p. 36, last para).

To turn next to the work of Cameron & Martin (1947), their theorem 1 emerges readily from the current paper via the implication:

$$\eta_p\text{-orthogonal expansion (1.22)} \;\&\; \eta_p\text{-Hermite connection (1.29)} \Longrightarrow \text{Fourier Hermite Series Theorem},$$

cf. 16.17, 16.18. Our paper sheds no light, however, on their earlier work on the evaluation of scalar-valued Wiener integrals $\int_{C(\mathbb{R})} F(x)w\,(\mathrm{d}x)$, for interesting functionals F, where $w(\cdot)$ is Wiener's probability measure over $C(\mathbb{R})$.

[6] His functions $f(\nu,\cdot)$ introduced in (82) are not simple (but σ-simple) and cannot be plugged into (76). Furthermore, the inequality in (83) is quite wrong.

Next came Kakutani's paper (1950). His theorem too is seen to emerge from the previous implication:

$$\eta_p\text{-orthogonal expansion (1.22)} \ \& \ \eta_p\text{-Hermite connection (1.29)} \implies \text{Kakutani's Isomorphism Theorem,}$$

once it is noted Kakutani's isometry W_p (defined in Kakutani (1950, p. 321)) is the inverse of ours, more precisely $(p!)^{-1/2}\mathbb{E}_{\eta_p} = W_p^{-1}$, cf. 16.7.

We now come to Ito's important paper (1951). Ingeniously, Ito defines a (measureless) Daniel integral $I_p(\cdot)$ on $L_2(\mathbb{R}^p)$ with values in a Gaussian–Hilbert space. By an outstanding *tour de force* he shows that $I_{p_1+\cdots+p_n}(\times_{i=1}^{p} f_i^{xp_i})$ is equal to the Hermite polynomial product on the RHS of our (1.29) when $f_1,\ldots,f_n$ in $L_2(\mathbb{R})$ are orthogonal. Then invoking the Cameron–Martin Theorem, he obtains the orthogonal expansion for x in the form $\sum_{p=0}^{\infty} I_p(f_p)$. The implication involved in his proof thus runs:

$$I_p\text{-Hermite connection} \ \& \ \text{Cameron–Martin Theorem} \implies I_p\text{-orthogonal expansion.}$$

This appeal to the Cameron–Martin theorem is dispensable, however, for it turns out that in fact $I_p = \mathbb{E}_{\eta_p}$. The demonstration is straightforward, though not trivial, cf. 13.21. Thus Ito's theorem is equivalent to (1.22).

Ito's designation of his I_p as a 'multiple Wiener integral' is a misnomer, since nothing akin to η_p or I_p appears in Wiener's 1938 paper. The fact that Wiener's objective in [W, § 12] was limited to the task of approximating a p-chaos by a p-polynomial chaos, together with the fact that the Cameron–Martin theorem is a crucial ingredient in Ito's proof, shows that the current designation 'The Wiener–Ito expansion' for the I_p-orthogonal expansion should give way to the more accurate '*the Ito–Cameron–Martin expansion*'. It must be recalled, however, that in a report (1942) entitled 'Response of a non-linear device to noise', Wiener initiated the use of multiple integrals of the Brownian motion to analyse nonlinear transducers, and this eventually did lead him to the 'Wiener–Ito' expansion, which he presented in his 1958 book, without use of the Cameron–Martin theorem. The Kakutani–Ito approach was further developed by Neveu (1968) and others (cf. Gross 1976).

While the results of Cameron–Martin, Kakutani and Ito mentioned above are deducible from ours, the converse is not true. Their theories cover at best only the easy η_p measure and its integration; the measure ξ_p, the inclusions between the spaces S_{ξ_p}, as in (1.21), the projection results and the ξ_p, η_p relationships (1.23), (1.27) are lost.

Different in this regard is the Memoir of Engel (1982): our important theorem that ξ_p is CA on $\mathcal{D}_p$ (§ 5) is a special case of his theorem 4.5 on the countable additivity of the product σ_p of random measures $\tau_1,\ldots,\tau_p$. Engel's τ_i over $\mathbb{R}$ are akin to our ξ of (1.12), except that the τ_i are not assumed to be Gaussian. Engel's proof of his theorem 4.5 is strange, and he does not address the question as to what the covariance $(\sigma_p(D),\sigma_q(E))$ might be. It is not clear that this question can be answered, for the restraints that Engel imposes on the τ_i, to compensate for their non-Gaussianness, might not be strong enough to provide an equality akin to Wiener's (75).

In view of this limitation of Engel's Memoir, it is of no help to us. To get our covariance equality, we would have to revert to Wiener's (75), and this would mean retracing the steps in §§ 3, 4.

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As should be evident from the discussion above, some discipline with regard to combinatorial issues is essential to pursue our goals. In the rest of this section, we explain our special notation and conventions governing such matters. More specific ancillary material is covered in the Appendices A, B, C.

(*j*) *Cartesian products*

Let $\ell, m, n \in \mathbb{N}_+$ and let $A \subseteq \mathbb{R}^\ell$, $B \subseteq \mathbb{R}^m$, $C \subseteq \mathbb{R}^n$ and $r := \ell + m + n$. Then, cf. 1.1(d),

$$A \times B \times C := \{f : f \in \mathbb{R}^{[1,r]} = \mathbb{R}^r, (f(1), \ldots, f(\ell)) \in A,$$
$$(f(\ell+1), \ldots, f(\ell+m)) \in B \quad \& \quad (f(\ell+m+1), \ldots, f(\ell+m+n)) \in C\}.$$

For instance, if A is a rectangle in $\mathbb{R}^2$ and B is an ellipsoid in $\mathbb{R}^3$, then

$$\begin{aligned} A \times B &= \{f : f \in \mathbb{R}^{[1,5]} \quad \& \quad (f(1), f(2)) \in A \quad \& \quad (f(3), f(4), f(5)) \in B\} \\ &= \{t : t \in \mathbb{R}^5 \ \& \ (t_1, t_2) \in A \ \& \ (t_3, t_4, t_5) \in B\}. \end{aligned}$$

More formally:

1.30. *Definition.* (Cartesian product) Let $n \in \mathbb{N}_+$ & $\forall i \in [1, n]$, $p_i \in \mathbb{N}_+$ & $A_i \subseteq \mathbb{R}^{p_i}$, and write $p_0 = 0$. Then

$$\begin{aligned} \bigtimes_{i=1}^{n} A_i &:= A_1 \times \cdots \times A_n \\ &= \{f : f \in \mathbb{R}^{[1, p_1 + \cdots + p_n]} \quad \& \quad \forall i \in [1, n], \\ &\qquad (f(p_0 + \cdots + p_{i-1} + 1), \ldots, f(p_0 + \cdots + p_{i-1} + p_i)) \in A_i\}. \end{aligned}$$

Thus with every such Cartesian product P is associated with an $r \in \mathbb{N}_+$ such that $P \subseteq \mathbb{R}^r$, namely, $r = p_1 + p_2 + \cdots + p_n$.

(*k*) *The restriction operator on* $\mathbb{R}^p$

1.31. *Definition.* Let $p \in \mathbb{N}_+$ & $\emptyset \neq M = \{i_1, \ldots, i_m\} \subseteq [1, p]$ & $1 \leqslant i_1, \ldots, i_m \leqslant p$. Then $\wp_M$ is the operator on $\mathbb{R}^p$ into $\mathbb{R}^m$, such that

$$\forall t \in \mathbb{R}^p \quad \& \quad \forall \alpha \in [1, m], \qquad [\wp_M(t)](\alpha) = t(i_\alpha).$$

Briefly, $\wp_M(t) = (t(i_1), t(i_2), \ldots, t(i_m)) \in \mathbb{R}^m$. For $\emptyset = M \subseteq [1, p]$, we define $\wp_M(t) = 0 \in \mathbb{R}^0$.

Note. Notice that $\forall t \in \mathbb{R}^p$, the domain of the function $\wp_M(t)$ is not M but $[1, m] = [1, \#M]$. Thus $\wp_M \neq \mathrm{Rstr.}_M$ and Range $\wp_M \not\subseteq \mathbb{R}^M$, but $\wp_M = \mathrm{Rstr.}_{[1,m]}$ and Range $\wp_M \subseteq \mathbb{R}^m$. Indeed

$$\wp_M \in \mathrm{L}(\mathbb{R}^p, \mathbb{R}^m).$$

Only the number $m := \#(M)$ is involved on the RHS. The set M contributes only to the values of the function $\wp_M(t)$.

1.32. *Definition.* (Induced restriction) (a) Let $\emptyset \neq M \subseteq [1, p]$. Then

$$\forall A \subseteq \mathbb{R}^p, \qquad \wp_M(A) = \{\wp_M(f) : f \in A\}.$$

(b) Let $m \in [1, p]$, $M = \{i_1, i_2, \ldots, i_m\} \subseteq [1, p]$ & $1 \leqslant i_1 < i_2 < \ldots < i_m \leqslant p$. Then

$$\forall A \subseteq \mathbb{R}^m, \qquad \wp_M^{-1}(A) := \{t : t \in \mathbb{R}^p \quad \& \quad \wp_M(t) \in A\},$$

i.e. $\wp_M^{-1}(A)$ is the M-cylinder in $\mathbb{R}^p$ with cross section $A \subseteq \mathbb{R}^m$.

The borderline case $M = \emptyset$ recurs in the paper. For the restriction $\wp_\emptyset$ it induces, one can easily check that

$$(1.33) \qquad \forall p \in \mathbb{N}_+ \quad \& \quad \forall A \subseteq \mathbb{R}^p, \qquad \wp_\emptyset(A) = \begin{cases} \{0\}, & \text{if } A \neq \emptyset, \\ \emptyset, & \text{if } A = \emptyset. \end{cases}$$

(*l*) *Intervals and their faces*

Let $p \in \mathbb{N}_+$ and $P^1, \ldots, P^p \subseteq \mathbb{R}$. Then we call $P := P^1 \times \cdots \times P^p$ an *interval* in $\mathbb{R}^p$. The following notation is very useful.

$$(1.34) \qquad \begin{cases} \forall \text{ intervals } P \text{ of } \mathbb{R}^p \ \& \ \forall M \subseteq [1,p], \text{ we write } P_M := \wp_M(P), \\ \text{and call } P_M \text{ the } M\text{-}hyperface \text{ (briefly, } M\text{-face) of } P. \end{cases}$$

We leave it to the reader to verify the following triviality:

1.35. Triviality. (On P_M) *Let $p \in \mathbb{N}_+$, $P := P^1 \times \cdots \times P^p$, where $P^1, \ldots, P^p \subseteq \mathbb{R}$, and let $M \subseteq [1,p]$. Then*
(*a*) *for $M = \emptyset$, $P_M = P_\emptyset = \{0\} = \mathbb{R}^0$;*
(*b*) *for $M \neq \emptyset$, say $M = \{i_1, \ldots, i_m\}$, $1 \leqslant i_1 < \ldots < i_m \leqslant p$, we have*

$$P_M = P^{i_1} \times \cdots \times P^{i_m} \subseteq \mathbb{R}^m.$$

From this result we see that P_M is an m-dimensional hyperface of the p-dimensional interval P.

(*m*) *Permutations, symmetry and symmetrization*

$$(1.36) \qquad \forall p \in \mathbb{N}_+, \qquad \mathrm{Perm}(p) := \{\phi : \phi \text{ is a permutation on } [1,p] \text{ onto } [1,p]\}.$$

1.37. *Definition.* Let $\phi \in \mathrm{Perm}(p)$. Then

$$\begin{aligned} &\forall t \in \mathbb{R}^p, && t^\phi := t \circ \phi = (t_{\phi(1)}, t_{\phi(2)}, \ldots, t_{\phi(q)}), \\ &\forall A \subseteq \mathbb{R}^p, && A^{\phi^{-1}} := \phi^{-1}(A) = \{t : t \in \mathbb{R}^p \ \& \ t^\phi \in A\}, \\ &\forall A \subseteq \mathbb{R}^p, && A \text{ is } symmetric \quad \text{iff} \quad \forall \phi \in \mathrm{Perm}(p), A^\phi = A. \end{aligned}$$

1.38. Triviality. *Let $\phi \in \mathrm{Perm}(p)$ & $A \subseteq \mathbb{R}^p$. Then*

$$t \in A^{\phi^{-1}} \Leftrightarrow t^\phi \in A; \qquad t \in A \Leftrightarrow t^\phi \in A^\phi; \qquad t^{\phi^{-1}} \in A \Leftrightarrow t \in A^\phi.$$

1.39. *Definition.* Let f be a function on $\mathbb{R}^p$ to $\mathbb{R}$. Then
(*a*) $\forall \phi \in \mathrm{Perm}(p)$ & $\forall t \in \mathbb{R}^p$, $f^\phi(t) := f(t^\phi) = f(t_{\phi(1)}, \ldots, t_{\phi(p)})$;
(*b*) the *symmetrization* $\tilde{f}$ of f is defined by

$$\tilde{f} := \frac{1}{p!} \sum_{\phi \in \mathrm{Perm}(p)} f^\phi \quad \text{on} \quad \mathbb{R}^p;$$

(*c*) f is called *symmetric* iff $\forall \phi \in \mathrm{Perm}(p)$, $f^\phi = f$;
f is called *antisymmetric* iff $\forall \phi \in \mathrm{Perm}(p)$, $f^\phi = (\operatorname{sgn} \phi) f$.

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The following propositions are obvious.

1.40. Proposition. *Let $p \in \mathbb{N}_+$. Then*
(*a*) $\forall A_1, \ldots, A_p \subseteq \mathbb{R}$ & $\forall \phi \in \mathrm{Perm}(p)$,

$$(A_1 \times \cdots \times A_p)^{\phi^{-1}} = A_{\phi(1)} \times \cdots \times A_{\phi(p)} \ \ \& \ \ \bigtimes_{i=1}^{p} A_i \text{ is symmetric } \Leftrightarrow A_1 = \cdots = A_p;$$

(*b*) $\mathcal{P}_p^{\mathrm{sym}} = \{A^p : A \in \mathcal{D}_1\}$, *cf. 1.9;*
(*c*) *the symmetric subsets of* $\mathbb{R}^p$ *form a* σ*-algebra over* $\mathbb{R}^p$*;*
(*d*) $\mathcal{R}_p^{\mathrm{sym}}$, $\mathcal{D}_p^{\mathrm{sym}}$, $\mathcal{B}_p^{\mathrm{sym}}$ *are respectively a ring, a* δ*-ring, and a* σ*-algebra over* $\mathbb{R}^p$*, cf. 1.9, (1.10).*

1.41. Proposition. *Let $p \in \mathbb{N}_+$. Then*
(*a*) $\forall f_1, \ldots, f_p$ *on* $\mathbb{R}$ & $\forall \phi \in \mathrm{Perm}(p)$,

$$(f_1 \times \cdots \times f_p)^{\phi^{-1}} = f_{\phi(1)} \times \cdots \times f_{\phi(p)} \quad \& \quad f_1 \times \cdots \times f_p \text{ is symmetric } \Leftrightarrow f_1 = \cdots = f_p;$$

(*b*) *the symmetric functions on* $\mathbb{R}^p$ *to* $\mathbb{R}$ *form a linear algebra with unit 1 that is closed in the pointwise convergence topology.*

Also obvious is the following:

$$(1.42) \qquad \begin{cases} \forall A \in \mathbb{R}^p \quad \& \quad \forall \phi \in \mathrm{Perm}(p), \quad (\chi_A)^\phi = \chi_{\phi^{-1}(A)}, \\ A \text{ is symmetric iff } \chi_A \text{ is symmetric.} \end{cases}$$

Slightly less obvious are the following results:

$$(1.43) \qquad \begin{cases} \text{If } B \in \mathcal{B}_p^{\mathrm{sym}}, \text{ then } \exists \text{ a sequence } (D_n)_{n=1}^\infty \text{ in } \mathcal{D}_p^{\mathrm{sym}} \\ \text{such that } D_n \uparrow B_n \text{ as } n \to \infty. \end{cases}$$

$$(1.44) \qquad \begin{cases} \text{If } f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1) \text{ is symmetric, then } \exists \text{ a sequence } (s_n)_{n=1}^\infty \text{ such} \\ \text{that } s_n \in \mathcal{S}(\mathcal{B}_p^{\mathrm{sym}}, \mathbb{R}) \ni s_n(\cdot) \to f(\cdot) \quad \& \quad |s_n(\cdot)| \leqslant |f(\cdot)| \text{ on } \mathbb{R}^p. \end{cases}$$

$$(1.45) \qquad \begin{cases} \text{Every symmetric } s \text{ in } \mathcal{S}(\mathcal{B}_p, \mathbb{R}) \text{ has a representation} \\ \sum_{k=1}^r b_k \chi_{B_k}, \ \text{where } B_k \in \mathcal{B}_p^{\mathrm{sym}}. \end{cases}$$

As for the symmetrization, we have

$$(1.46) \qquad \forall f \text{ on } \mathbb{R}^p \text{ to } \mathbb{R} \quad \& \quad \forall t \in \mathbb{R}^p, \qquad |\tilde{f}(t)| \leqslant |f|\tilde{\ }(t).$$

Part I. Chaotic measure theory

2. The Venn expansion and proof of Wiener's equality (75)

Since Wiener's equality (75) in the format 1.18 is central to the entire paper, we shall indicate its proof. This depends on two lemmas. The first of these lemmas, which we shall take for granted, concerns products of random variables obtained from *independent* normally distributed random variables:

2.1. Lemma. *Let* (i) $x_1, \ldots, x_r$ *be* r *independent normally distributed random variables over* $(\Omega, \mathcal{A}, \mathbb{P})$ *such that*

$$\mathbb{E}_{\mathbb{P}}(x_i) = 0 \quad \& \quad \mathbb{E}_{\mathbb{P}}(x_i^2) = \delta_i^2 > 0.$$

(ii) $(y_k)_{k=1}^n$ *be a sequence with range* $\subseteq \{x_1, \ldots, x_r\}$ *and* n_i *be the frequency of* x_i *in* $(y_k)_{k=1}^n$ *(thus* $0 \leqslant n_i$ & $\sum_{i=1}^r n_i = n$*). Then*

(*a*) $\forall$ *odd* n,

$$\mathbb{E}_{\mathbb{P}}\left(\prod_{k=1}^n y_k\right) = 0;$$

(*b*) $\forall$ *even* $n = 2m$, *cf. 1.1(i) and 1.16*(*a*), (*b*),

$$\mathbb{E}_{\mathbb{P}}\left(\prod_{k=1}^n y_k\right) = \prod_{i=1}^r \alpha_{n_i}\delta_i^{n_i} = \sum_{\pi \in \Pi_{[1,n]}} \prod_{\Delta \in \pi} (y_{\min \Delta}, y_{\max \Delta})_{\mathcal{L}_2}.$$

In (*b*) the second equality rests on the fact that the product on the RHS is non-zero only for those $\pi \in \Pi_{[1,n]}$ for which for each cell $\Delta \in \pi$, we have $y_{\min \Delta} = y_{\max \Delta} =$ same x_i, and therefore $(y_{\min \Delta}, y_{\max \Delta})_{\mathcal{L}_2} = \delta_i^2$.

The second lemma specifies the expansion implicit in the Venn diagram of the n edges of an interval in $\mathbb{R}^n$:

2.2. Lemma. (Venn expansion) *Let*

(i) $n \in \mathbb{N}_+$, $A_1, \ldots, A_n \subseteq \mathbb{R}$ & $A = \times_{i=1}^n A_i$,

(ii) $(B_1, \ldots, B_{2^n-1})$ *be the sequence of cells in the Venn diagram of the family* $\{A_1, \ldots, A_n\}$ *(in any order) that are inside* $\bigcup_{i=1}^n A_i$*; thus*

$$\emptyset \subseteq B_j \subseteq \mathbb{R} \quad \& \quad \bigcup_{j=1}^{2^n-1} B_j = \bigcup_{i=1}^n A_i,$$

(iii) $\forall i \in [1,n]$, $N_i := \{j : j \in [1, 2^n - 1]$ & $B_j \subseteq A_i\}$ *(so that each* $A_i = \bigcup_{j \in N_i} B_j$*).*

Then $\forall i, j, k, \ldots \in [1,n], i, j, k, \ldots$ *distinct, we have*

(*a*) $\#N_i = 2^{n-1}$, $\#(N_i \cap N_j) = 2^{n-2}, \ldots, \#(\cap_{i=1}^p N_i) = 1$;

(*b*) $A_i = \bigcup_{j \in N_i} B_j$ & $B_j \| B_{j'}$ *for* $j \neq j'$;

(*c*) $A = \bigcup_{j_1 \in N_1} \cdots \bigcup_{j_n \in N_n} (B_{j_1} \times \cdots \times B_{j_n})$, *and the* $(2^{n-1})^n = 2^{n(n-1)}$ *intervals on the RHS are* $\|$.

Proof. (Gist) (*a*) is obvious. As for (*b*), N_i is the set of subscripts j for which B_j is the jth Venn cell included in A_i, and so $A_i = \bigcup_{j \in N_i} B_j$. The B_j, being Venn cells are of course $\|$. (*c*) follows on substituting from the last equality in $A = \times_{i=1}^n A_i$, and simplifying. ∎

Gist of proof of theorem 1.18. Let the notation be as in 2.2. We need prove part (*b*) only for n even. Then $\forall i \in [1,n]$, by 2.2(*b*),

$$\xi(A_i) = \sum_{j \in N_i} \xi(B_j) = \sum_{j=1}^{2^n} \chi_{N_i}(j)\xi(B_j). \tag{1}$$

Hence by the generalized distribution law,

$$\prod_{i=1}^n \xi(A_i) = \sum_{j_1=1}^{2^n} \cdots \sum_{j_n=1}^{2^n} \chi_{N_1}(j_1)\xi(B_{j_1}) \cdots \chi_{N_n}(j_n)\xi(B_{j_n})$$
$$= \sum_{j_1=1}^{2^n} \cdots \sum_{j_n=1}^{2^n} b_{j_1 \ldots j_n} \xi(B_{j_1}) \cdots \xi(B_{j_n}),$$

where $b_{j_1\ldots j_n} := \chi_{N_1}(j_1)\cdots\chi_{N_n}(j_n)$. Thus

$$(2)\qquad \mathbb{E}_{\mathbb{P}}\left\{\prod_{i=1}^{n}\xi(A_i)\right\} = \sum_{j_1=1}^{2^n}\cdots\sum_{j_n=1}^{2^n} b_{j_1\ldots j_n}\mathbb{E}_{\mathbb{P}}\{\xi(B_{j_1})\cdots\xi(B_{j_n})\}.$$

Now any two of the $B_{j_1}\cdots B_{j_n}$ are either identical or $\|$. Hence letting $y_k := \xi(B_{j_k})$, the sequence $(y_k)_{k=1}^{n}$ satisfies the premises of 2.1. Hence by 2.1(b),

$$(3)\qquad \mathbb{E}_{\mathbb{P}}\{\xi(B_{j_1})\cdots\xi(B_{j_n})\} = \sum_{\pi\in\Pi_{[1,n]}}\prod_{\Delta\in\pi}(\xi(B_{j_{\min\Delta}}),\xi(B_{j_{\max\Delta}}))_{\mathcal{L}_2}.$$

It follows from (3) that

$$\mathrm{RHS}(2) = \sum_{j_1=1}^{2^n}\cdots\sum_{j_n=1}^{2^n} b_{j_1\ldots j_n}\sum_{\pi\in\Pi_{[1,n]}}\prod_{\Delta\in\pi}(\xi(B_{j_{\min\Delta}}),\xi(B_{j_{\max\Delta}}))_{\mathcal{L}_2}.$$

The $\sum_{\pi\in\Pi}$ can be brought out. Hence to get 1.18(b) we need only show that

$$\text{(I)}\qquad \forall\pi\in\Pi_{[1,n]},\qquad \sum_{j_1=1}^{2^n}\cdots\sum_{j_n=1}^{2^n} b_{j_1\ldots j_n}\prod_{\Delta\in\pi}(\xi(B_{j_{\min\Delta}}),\xi(B_{j_{\max\Delta}}))_{\mathcal{L}_2} = \prod_{\Delta\in\pi}(\xi(A_{j_{\min\Delta}}),\xi(A_{j_{\max\Delta}}))_{\mathcal{L}_2}.$$

We leave to the reader the cumbersome but routine proof of (I), based on expanding the inner products on the RHS of (I) by means of (1), and simplifying. ■

Recall, cf. proposition 1.40(b), that

$$\forall n\in\mathbb{N}_+,\qquad P\in\mathcal{P}_n^{\mathrm{sym}}\quad\Longleftrightarrow\quad\exists A\in\mathcal{D}_1\ni P=A^n,$$

i.e. an interval in $\mathcal{P}_n$ is symmetric iff it is a hypercube A^n, $A\in\mathcal{D}_1$. This justifies the following terminology:

2.3. *Definition.* Let $n\in\mathbb{N}_+$ and $P\in\mathcal{P}_n$. We say that

(a) P is *hyposymmetric* iff P is a Cartesian product of $\|$ symmetric intervals, i.e. iff $\exists r\in\mathbb{N}_+$, $\exists\|A_1,\ldots,A_r\in\mathcal{D}_1$ & $\exists n_1,\ldots,n_r\in\mathbb{N}_+$ with $n_1+\cdots+n_r=n$ such that $P=\times_{i=1}^{r}A_i^{n_i}$.

(b) P is *permutation hyposymmetric* iff $\exists\phi\in\mathrm{Perm}(p)\ni P^{\phi}$ is hyposymmetric.

Example. Let $A_1,A_2,A_3\in\mathcal{D}_1$ be $\|$. Then

$$P := A_1\times A_3\times A_1\times A_2\times A_2\times A_1\times A_3\times A_3\in\mathcal{P}^8$$

is permutation hyposymmetric, since there exists a $\phi\in\mathrm{Perm}(8)$ such that $P^{\phi}=A_1^3\times A_2^2\times A_3^3$, is hyposymmetric.

In this terminology the Venn expansion in lemma 2.2(c) can be recapitulated as follows:

2.4. Theorem. *Let $n\in\mathbb{N}_+$, $A_1,\ldots,A_n\subseteq\mathbb{R}$ & $A:=\times_{i=1}^{n}A_i$. Then $\exists m=2^{n(n-1)}\|$ permutation hyposymmetric intervals $\bar{B}_1\cdots\bar{B}_m\subseteq\mathbb{R}^n$ such that $A=\bigcup_{k=1}^{m}\bar{B}_k$.*

Proof. With the notation used in 2.2, we see that the $j_1,j_2,\ldots,j_n$, which index an interval $\bar{B}_{j_1\ldots j_n} := B_{j_1}\times B_{j_2}\times\cdots\times B_{j_n}\in\mathcal{P}_n$ on the RHS of 2.2(c), need not be distinct. In other words, two or more sides of $\bar{B}_{j_1\ldots j_n}$ can be the same. By a suitable permutation ϕ in Perm(n), we can bring the repeated subscripts together, thereby obtaining an interval $(\bar{B}_{j_1\ldots j_n})^{\phi} = C_1^{\nu_1} \times C_2^{\nu_2} \times \cdots \times C_r^{\nu_r}$, where $r \in [1, n]$ and $\nu_1, \ldots, \nu_r \in \mathbb{N}_+$ & $\nu_1 + \cdots + \nu_r = n$. In this $C_1, C_2, \ldots, C_r \subseteq \mathcal{D}_1$ are $\|$, since $B_1 \cdots B_{2^r-1}$ are $\|$. Thus $(\bar{B}_{j_1\ldots j_n})^{\phi}$ is hyposymmetric and $\bar{B}_{j_1\ldots j_n}$ is permutation hyposymmetric, by 2.3(b).

Thus 2.2(c) reduces to

$$A = \bigcup_{j_1 \in N_1} \cdots \bigcup_{j_n \in N_n} \bar{B}_{j_1\ldots j_n},$$

i.e. A is a union of $m = 2^{n(n-1)}\|$ permutation hyposymmetric intervals. ■

Theorem 2.4 has an important bearing on the structure of the measures ξ_p on $\mathcal{P}_p$. We first note that from proposition 1.40(a) and the definition (1.13) of ξ_p, it follows readily that

(2.5) $$\forall P \in \mathcal{P}^p \quad \& \quad \forall \phi \in \text{Perm}(p), \qquad \xi_p(P^{\phi}) = \xi_p(P).$$

We now assert the following result on the structure of $\xi_p(P)$.

2.6. Corollary. *Let $p \in \mathbb{N}_+$ and $P \in \mathcal{P}_p$. Then $\exists r = 2^{p(p-1)}\|$ hyposymmetric intervals $Q_1, \ldots, Q_r \in \mathcal{P}_p$ such that*

$$\xi_p(P) = \sum_{k=1}^{r} \xi_p(Q_k).$$

Proof. By 2.4, $\exists r = 2^{p(p-1)}\|$ permutation hyposymmetric intervals $\bar{Q}_1, \ldots, \bar{Q}_r$ in $\mathcal{P}_p$ such that $P = \bigcup_{k=1}^{r} \bar{Q}_k$. Since, cf. (1.14), ξ_p is FA on $\mathcal{P}_p$, it follows that

(1) $$\xi_p(P) = \sum_{k=1}^{r} \xi_p(\bar{Q}_k).$$

But since $\bar{Q}_k$ is permutation hyposymmetric, therefore $\exists$ a hyposymmetric interval $Q_k \in \mathcal{P}_p$ and $\exists$ a permutation $\phi_k \in \text{Perm}(p)$ such that $\bar{Q}_k = Q_k^{\phi_k}$, whence by (2.5), $\xi_p(\bar{Q}_k) = \xi_p(Q_k^{\phi_k}) = \xi_p(Q_k)$. Hence (1) reduces to the desired equality. ■

3. Wiener's p-homogeneous chaotic measure on the pre-ring $\mathcal{P}_p$ of intervals

In (1.13) we defined for any $p \in \mathbb{N}_+$, the p-homogeneous chaotic measure ξ_p on the pre-ring $\mathcal{P}_p$ of intervals of $\mathbb{R}^p$ with edges in $\mathcal{D}_1$ and noted that $\xi_p \in \text{FA}(\mathcal{P}_p, \mathcal{L}_2)$. For completeness we have to include the trivial case $p = 0$, and to this we first attend:

3.1. *Extension to the case* $p = 0$. Since, cf. 1.1(d), $\mathbb{R}^0 = \{0\}$, we define ξ_0 on the trivial algebra $\mathcal{A}_0 := \{\emptyset, \mathbb{R}^0\}$ over $\mathbb{R}^0$ by

$$\xi_0(\emptyset) := 0 \in \mathcal{L}_2 \quad \& \quad \xi_0(\mathbb{R}^0) = \xi_0(\{0\}) := 1(\cdot) \in \mathcal{L}_2,$$

where $1(\cdot)$ is the constant-valued function with value 1. Likewise, we define $\ell_0(\cdot)$ on $\mathcal{A}_0$ by

$$\ell_0(\emptyset) := 0 \in \mathbb{R} \quad \& \quad \ell_0(\mathbb{R}^0) = \ell_0(\{0\}) := 1 \in \mathbb{R}.$$

Trivially, $\xi_0 \in \mathrm{CA}(\mathcal{A}_0, \mathcal{L}_2)$, $\ell_0 \in \mathrm{CA}(\mathcal{A}_0, \mathbb{R}_{0+})$. Note that every real (or complex-) valued f on $\mathbb{R}^0$ is $\mathcal{A}_0$-measurable, that

$$\int_{\mathbb{R}^0} f(t)\ell_0(\mathrm{d}t) = f(0),$$

and that

$$\left\{\int_{\mathbb{R}^0} f(t)\xi_0(\mathrm{d}t)\right\}(\cdot) = f(0) \quad \text{on} \quad \Omega.$$

It follows from 1.14 and 3.1 that

$$(3.2) \qquad \forall p \in \mathbb{N}_{0+}, \qquad \xi_p \in \mathrm{FA}(\mathcal{P}_p, \mathcal{L}_2).$$

The following lemma on the raw moments of $\xi_p(P)$ is important:

3.3. Lemma. *Let $p \in \mathbb{N}_{0+}$. Then $\forall P \in \mathcal{P}_p$ and $\forall r \in \mathbb{N}_+$, $\xi_p(P) \in \mathcal{L}_r := L_r(\Omega, \mathcal{A}, P; \mathbb{R})$ &*

$$|\xi_p(P)|^r_{\mathcal{L}_r} := \mathbb{E}_{\mathbb{P}}\{|\xi_p(P)|^r\} \leqslant \gamma_{rp} \cdot [\ell_p(P)]^{r/2},$$

where

$$\gamma_n := \begin{cases} \alpha_n, & \text{if } n \text{ is even}, \\ 2^{[n/2]}[n/2]\surd(2/\pi), & \text{if } n \text{ is odd}. \end{cases}$$

Thus, $\xi_p(P)$ has finite (absolute raw) moments of all orders $r \in \mathbb{N}_+$. In particular

$$\xi_p(P) \in \mathcal{L}_2 \quad \& \quad |\xi_p(P)|^2_{\mathcal{L}_2} \leqslant \alpha_{2p}\ell_p(P).$$

Proof. Let $P := P^1 \times \cdots \times P^p \in \mathcal{P}_p$ and $r \in \mathbb{N}_+$. In the generalized Schwartz inequality for non-negative random variables $X_1, \ldots, X_p$:

$$(1) \qquad \mathbb{E}_{\mathbb{P}}\left(\prod_{i=1}^p X_i\right) \leqslant \left\{\prod_{i=1}^p \mathbb{E}_{\mathbb{P}}(X_i^p)\right\}^{1/p},$$

take $X_i = |\xi_1(P^i)|^r$. Then, since $\xi_1(P^i)$ is $(0, \ell_1(P^1))$ normally distributed, we have for its rpth absolute raw moment:

$$\mathbb{E}_{\mathbb{P}}(X_i^p) = \mathbb{E}_{\mathbb{P}}\{|\xi(P^i)|^{rp}\} = \begin{cases} \alpha_{rp}\ell_1(P^i)^{rp/2}, & \text{if } rp \text{ is even}, \\ 2^{[rp/2]}[rp/2]!\surd(2/\pi)\cdot\ell_1(P^i)^{rp/2}, & \text{if } rp \text{ is odd}, \end{cases}$$
$$=: \gamma_{rp}\ell_1(P^i)^{rp/2}.$$

It easily follows that $\mathrm{RHS}(1) = \gamma_{rp} \cdot \ell_p(P)^{r/2}$. Next,

$$\prod_{i=1}^p X_i = \prod_{i=1}^p |\xi_1(P^i)|^r = |\xi_1(P^1)\cdots\xi_1(P^p)|^r = |\xi_p(P)|^r.$$

Thus (1) reduces to

$$\mathbb{E}_{\mathbb{P}}\{|\xi_p(P)|^r\} \leqslant \gamma_{rp}\ell_p(P)^{r/2}.$$

■

From (3.2) and vector-measure theory we know that ξ_p has an FA extension to $\mathcal{R}_p$. Denoting this extension by the same symbol ξ_p, we have

$$(3.4) \qquad \forall p \in \mathbb{N}_{0+}, \qquad \xi_p \in \mathrm{FA}(\mathcal{R}_p, \mathcal{L}_2).$$

The inequality in 3.3 notwithstanding, ξ_p is not absolutely continuous with respect to ℓ_p (in symbols, $\xi_p \not\ll \ell_p$). This is best shown by first showing that $\xi_2 \not\ll \ell_2$, and then adapting this result for $p \geqslant 3$. For $p = 2$, we apply Wiener's equality 1.18(b) taking $p = 4$ and $P^1 = P^2 = A \in \mathcal{D}_1$ and $P^3 = P^4 = B \in \mathcal{D}_1$; thus

$$(\xi_2(A^2), \xi_2(B^2)) = \mathbb{E}_{\mathbb{P}}\left\{\prod_{i-1}^{p} \xi_1(P^i)\right\} = \ell_1(A)\ell_1(B) + 2\{\ell_1(A\cap B)\}^2.$$

From this we easily get the following estimate for the ξ_2 measure of finite unions of $\|$ squares:

$$\text{(3.5)}\qquad \begin{cases} A_1,\ldots,A_n \in \mathcal{D}_1 \quad \& \quad A_1,\ldots,A_n \quad \text{are} \quad \| \\ \qquad\qquad \Longrightarrow \left|\xi_2\left(\bigcup_{i=1}^{n} A_i^2\right)\right|^2 \geqslant \left[\ell_1\left(\bigcup_{i=1}^{n} A_i\right)\right]^2. \end{cases}$$

This in turn suggests the following example to refute the absolute continuity of ξ_2 with respect to ℓ_2.

3.6. *Example.* ($\xi_2 \not\ll \ell_2$) We consider the binary subintervals of $(0,1]$:

$$\begin{array}{cccc} (0,\frac{1}{2}) & [\frac{1}{2},1] & & \\ (0,\frac{1}{4}) & (\frac{1}{4},\frac{1}{2}] & (\frac{1}{2},\frac{3}{4}] & (\frac{3}{4},1] \\ \vdots & & & \\ (0,1/2^n] & (1/2^n,2/2^n] & \ldots & ((2^n-1)/2^n,1] \\ \vdots & & & \end{array}$$

Denoting those on the nth row by $A_{n,1}, A_{n,2}, \ldots, A_{n,2^n}$, we see that

$$\forall n \geqslant 1, \qquad \ell_1\left(\bigcup_{k=1}^{2^n} A_{n,k}\right) = \ell_1(0,1] = 1.$$

Hence letting $R_n := \bigcup_{k=1}^{2^n}(A_{n,k})^2$, we see from (3.5) that

$$\text{(1)}\qquad \forall n \in \mathbb{N}_+, \qquad |\xi_2(R_n)| \geqslant 1.$$

But

$$\forall n \in \mathbb{N}_+, \qquad \ell_2(R_n) = \sum_{k=1}^{2^n} \ell_2(A_{n,k}^2) = \sum_{k=1}^{2^n}(1/2^n)^2 = 1/2^n.$$

Hence $\ell_2(R_n) \to 0$, as $n \to \infty$. Thus $\xi_2 \not\ll \ell_2$. ■

The adaptation of this example to any $p \geqslant 3$, by the consideration of cylinders based on the squares $A_{n,k}^2$, hinges on the following lemma:

3.7. Lemma. *Let $A, B, C \in \mathcal{D}_1$ and $p \geqslant 3$. Then*

$$(\xi_p(A^2 \times C^{p-2}), \xi_p(B^2 \times C^{p-2}))_{\mathcal{L}_2} \geqslant \alpha_{2p-4}(\xi_2(A^2), \xi_2(B^2))_{\mathcal{L}_2} \cdot \ell_1(C)^{p-2}.$$

Proof. Let a stand for the LHS. Then by (1.13),

$$a = \mathbb{E}_{\mathbb{P}}\{\xi_1(A)^2\xi_1(C)^{p-2}\xi_1(B)^2\xi_1(C)^{p-2}\} = \mathbb{E}_{\mathbb{P}}\{\xi_1(A)^2\xi_1(B)^2\xi_1(C)^{2p-4}\}$$

$$\text{(1)}\qquad = \sum_{\pi\in\Pi_{[1,2P]}} \prod_{\Delta\in\pi} \ell_1\{P(\Delta)\} \quad \text{by 1.19}(b),$$

where $P := A \times A \times B \times B \times C \times \cdots \times C$, with $(2p-4)C$'s, and so $P \in \mathcal{P}_{2p}$.

Now given any $\pi_1 \in \Pi_{[1,4]}$ and any $\pi_2 \in \Pi_{[5,2p]}$, we have $\pi := \pi_1 \cup \pi_2 \in \Pi_{[1,2p]}$. For this π,

(2) $$\prod_{\Delta\in\pi} \ell_1\{P(\Delta)\} = \prod_{\Delta_1\in\pi_1} \ell_1\{Q(\Delta_1)\} \cdot \prod_{\Delta_2\in\pi_2} \ell_1\{R(\Delta_2)\},$$

where $Q = A \times A \times B \times B \in \mathcal{P}_4$ & $R = C^{2p-4} \in \mathcal{P}_{2p-4}$. Denoting by $\Pi^0_{[1,2p]}$ the subclass of $\Pi_{[1,2p]}$ made of such decomposable π, we see from (2) that

$$\sum_{\pi\in\Pi^0_{[1,2p]}} \prod_{\Delta\in\pi} \ell_1\{P(\Delta)\} = \sum_{\pi_1\in\Pi_{[1,4]}} \prod_{\Delta_1\in\pi_1} \ell_1\{Q(\Delta_1)\} \cdot \sum_{\pi_2\in\Pi^0_{[5,2p]}} \prod_{\Delta_2\in\pi_2} \ell_1\{R(\Delta_2)\}$$

(3) $$= \mathbb{E}_{\mathbb{P}}\{\xi_4(Q)\} \cdot \mathbb{E}_{\mathbb{P}}\{\xi_{2p-4}(R)\}.$$

Since $\Pi_{[1,2p]} \supset \Pi^0_{[1,2p]}$, it follows from (1) and (3) that

(4) $$a \geqslant \mathbb{E}_{\mathbb{P}}\{\xi_4(Q)\} \cdot \mathbb{E}_{\mathbb{P}}\{\xi_{2p-4}(R)\}.$$

But

(5) $$\mathbb{E}_{\mathbb{P}}\{\xi_4(Q)\} = \mathbb{E}_{\mathbb{P}}\{\xi_2(A^2)\cdot\xi_2(B^2)\} = (\xi_2(A^2), \xi_2(B^2)),$$

and by 1.19(d),

(6) $$\mathbb{E}_{\mathbb{P}}\{\xi_{2p-4}(R)\} = \mathbb{E}_{\mathbb{P}}\{\xi_1(C)^{2p-4}\} = \alpha_{2p-4}\ell_1(C)^{p-2}.$$

Substituting from (5) and (6) in (4), we get the desired inequality. ■

3.8. *Example.* ($\xi_p \not\ll \ell_p$ for $p \geqslant 2$) Let $p \geqslant 3$ be fixed, and let

$$\forall n \in \mathbb{N}_+, \qquad R_n = \bigcup_{k=1}^{2^n} (A_{n,k} \times [0,1]^{p-2}),$$

where the $A_{n,k}$ are as in 3.6. Since for $k \in [1, 2^n]$, the sets $A^2_{n,k} \times [0,1]^{p-2}$ are $\|$, we have, writing $C = [0,1]$,

$$\xi_p(R_n) = \sum_{k=1}^{2^n} \xi_p(A^2_{n,k} \times C^{p-2}).$$

Hence by 3.7,

(1) $$|\xi_p(R_n)|^2_{\mathcal{L}_2} \geqslant \sum_{i=1}^{2^n}\sum_{j=1}^{2^n} \alpha_{2p-4}(\xi_2(A^2_{n,i}), \xi_2(A^2_{n,j}))_{\mathcal{L}_2} \cdot \ell_1(C)^{p-2}.$$

But by the inequality 3.6(1),

$$\mathrm{RHS}(1) = \alpha_{2p-4}\left|\xi_2\left(\bigcup_{i=1}^{2^n} A^2_{n,i}\right)\right|^2 \geqslant \alpha_{2p-4}.$$

Thus by (1), $\forall n \in \mathbb{N}_+$, $|\xi_p(R^n)|^2 \geqslant \alpha_{2p-4}$. But again

$$\ell_p(R^n) = \sum_{k=1}^{2^n} \ell_p(A^2_{n,k} \times C^{p-2}) = \sum_{k=1}^{2^n} \ell_1(A_{n,k})^2 \ell_{p-2}(C^{p-2}) = 1/2^n.$$

Hence $\ell_p(R_n) \to 0$, as $n \to \infty$. Thus $\xi_p \not\ll \ell_p$. ■

Example 3.8 reveals the complexity of the measure ξ_p *vis-à-vis* the measure ξ_1, of which it is the product. The complexity of the ξ_p is more fully revealed by the cross covariance formulae for $(\xi_p(P), \xi_q(Q))$, for $P \in \mathcal{P}_p$ and $Q \in \mathcal{P}_q$, to which we now turn.

To obtain this cross-covariance we apply Wiener's equality 1.18 to the sequence of the edges of P, Q:

(1) $$(P^1, P^2, \ldots, P^p, Q^1, Q^2, \ldots, Q^q).$$

From 1.18(a), we see at once that

(3.9) $$\forall P \in \mathcal{P}_p \quad \& \quad \forall Q \in \mathcal{P}_q, \qquad (\xi_p(P), \xi_q(Q))_{\mathcal{L}_2} = 0, \text{ if } p+q \text{ is odd.}$$

To deal with the case of even $p + q = 2r$, we note that since

$$(\xi_p(P), \xi_q(Q))_{\mathcal{L}_2} = (\xi_q(Q), \xi_p(P))_{\mathcal{L}_2},$$

we can assume, without loss of generality, that $q \leqslant p$. After applying 1.18(b) to the sequence $(A_k)_{r=1}^{2r}$ given by (1), we have to suitably partition the class $\Pi_{[1,2r]}$ into subclasses of partitions in order to express the answer in terms of the intervals P, Q and their hyperfaces.

Recall that for $P = \times_{i=1}^{n} P^i \subseteq \mathbb{R}^n$, the set $M \subseteq [1, n]$ *determines the hyperface of P generated by the sides P^i of P for which $i \in M$.* As we saw in 1.35, this hyperface is

$$P_M := \mathop{\times}_{i \in M} P^i \subseteq \mathbb{R}^m, \quad \text{where} \quad m = \#M \quad \& \quad P_\emptyset = \{0\} = \mathbb{R}^0.$$

Recall the notations used in 1.19(c). They are worth standardizing:

(3.10) $$\begin{cases} \text{For } P = \mathop{\times}_{i=1}^{n} P^i \in \mathcal{P}_n \ \& \ \Delta = \{i, j\} \ \& \ \pi \in \Pi_M, \\ \text{where } M \subseteq [1, n] \text{ has even cardinality } 2m, \\ \qquad P(\Delta) := P^i \cap P^j \quad \& \quad P(\pi) := \mathop{\times}_{\Delta \in \pi} P(\Delta) \in \mathcal{P}_m. \\ \text{When } m = 0, \text{ i.e. } M = \emptyset, \text{ we define } P(\pi) = P(\Delta) = \{0\}. \end{cases}$$

For the Lebesgue measure of $P(\pi)$, we write, cf. 1.16(d):

(3.11) $$\forall n \in \mathbb{N}_+, \quad \forall m \in [1, [n/2]] \quad \& \quad \forall \pi \in \Pi_m^n, \quad a_\pi^n(P) := \ell_m\{P(\pi)\}.$$

For $m = 0$, cf. 3.1, $a_\pi^n(P) = a_\emptyset^n(P) = 1$. For $m \geqslant 1$, $a_\pi^n(P) = \prod_{\Delta \in \pi} \ell_1\{P(\Delta)\}$. Hence by 1.19($b$).

(3.12) $$\begin{cases} \forall \text{ even } n \text{ and } r = n/2, \\ \displaystyle\sum_{\pi \in \Pi_{n/2}^n} a_\pi^n(P) := \sum_{\pi \in \Pi_{[1,n]}} \prod_{\Delta \in \pi} \ell_1\{P(\Delta)\} = \mathbb{E}_{\mathbb{P}}\{\xi_n(P)\}. \end{cases}$$

With the notation (3.11), Wiener's equality can be formulated in the following way, which yields the desired covariances:

3.13. Theorem. (Wiener's equality for the cross-covariance) *Let*

(i) *$p, q \in \mathbb{N}_+$ be such that $p + q = 2r$ is even and $q \leqslant p$,*

(ii) *$P \in \mathcal{P}_p$ & $Q \in \mathcal{P}_q$.*

Then, with the notation 1.16 and (3.11),

(*a*) *when* $q < p$, $(\xi_p(P), \xi_q(Q))_{\mathcal{L}_2} = \sum_{k=0}^{[q/2]} \Gamma_k^{pq}(P,Q) \in \mathbb{R}_{0+}$, *where*

$$\Gamma_0^{pq}(P,Q) := \sum_{\phi\in\mathrm{Perm}(q)} \sum_{\pi\in\Pi^p_{(p-q)/2}} a^p_\pi(P)\ell_q(P_{M'_\pi}\cap Q^\phi),$$

& *for* $k \in [1, [q/2]-1]$,

$$\Gamma_k^{pq}(P,Q) := \sum_{\phi\in\mathrm{Perm}(q-2k)} \sum_{\pi_1\in\Pi^p_{\frac{1}{2}(p-q)+k}} \sum_{\pi_2\in\Pi^q_k} a^p_{\pi_1}(P)a^q_{\pi_2}(Q)\ell_{q-2k}\{(P_{M'_{\pi_1}})\cap(Q_{M'_{\pi_2}})^\phi\},$$

where $M'_{\pi_1} := [1,p]\backslash M_{\pi_1}$, $M'_{\pi_2} := [1,q]\backslash M_{\pi_2}$, *and*

$$\Gamma^{pq}_{[q/2]}(P,Q) := \begin{cases} \sum_{\pi_1\in\Pi^p_{[p/2]}} a^p_{\pi_1}(P) \sum_{\pi_2\in\Pi^q_{[q/2]}} a^q_{\pi_2}(Q) = \mathbb{E}_{\mathbb{P}}\{\xi_p(P)\}\cdot\mathbb{E}_{\mathbb{P}}\{\xi_q(Q)\}, & q \text{ even},\\ \sum_{\pi_1\in\Pi^p_{[p/2]}} \sum_{\pi_2\in\Pi^q_{[q/2]}} a^p_{\pi_1}(P)a^q_{\pi_2}(Q)\ell_1(P_{M'_{\pi_1}}\cap Q_{M'_{\pi_2}}), & q \text{ odd}.\end{cases}$$

(*b*) *when* $q = p$, $(\xi_q(P), \xi_q(Q))_{\mathcal{L}_2} = \sum_{k=0}^{[q/2]} \Gamma_k^{qq}(P,Q)$, *where now*

$$\Gamma_0^{qq}(P,Q) := \sum_{\phi\in\mathrm{Perm}(q)} \ell_q(P\cap Q^\phi),$$

and $\forall k \in [1,[q/2]]$, $\Gamma_k^{pp}(P,Q)$ *is given by the formula in* (*a*) *with* p *replaced by* q.

Proof. We apply 1.18(*b*) to the sets

$$(1)\qquad \begin{cases} A_1, A_2, \ldots, A_p, A_{p+1}\cdots A_{p+q} = A_{2r},\\ \text{where}\\ \forall i\in[1,p],\quad A_i = P^i \quad \&\quad \forall j\in[1,q],\quad A_{p+j} = Q^j.\end{cases}$$

Call a cell $\Delta \in \pi \in \Pi_r^{p+q}$ 'good' iff $1 \leqslant \min\Delta \leqslant p < \max\Delta \leqslant 2r$. Let

$$\Pi^{p+q}_{r,\rho} := \{\pi : \pi\in\Pi^{p+q}_r \text{ has } \rho \text{ 'good' cells}\}.$$

We then have the $\|$ decomposition

$$\Pi^{p+q}_r = \bigcup_{k=0}^{[q/2]} \Pi^{p+q}_{r,q-2k},$$

and the equality in 1.18(*b*) becomes

$$(2)\qquad (\xi_p(P), \xi_p(Q))_{\mathcal{L}_2} = \sum_{k=0}^{[q/2]} \sum_{\pi\in\Pi^{p+q}_{r,q-2k}} \prod_{\Delta\in\pi} \ell_1\{A(\Delta)\}.$$

Now let $1 \leqslant k < [q/2]$. Then to each $\pi \in \Pi^{p+q}_{r,q-2k}$ corresponds first , sets $M \subseteq [1,p]$, $L \subseteq [p+1, p+q]$ such that $\#M = (p-q)+2k$, $\#L = 2k$, and therefore $\#M' = \#L' =$

$q-2k$, where $M' := [1,p]\backslash M$, $L' := [p+1,p+q)\backslash L$; second, a one–one function ψ on M' onto L'; and third, partitions $\pi_1 \in \Pi_M$, $\pi_3 \in \Pi_L$, such that

$$\pi = \pi_1 \cup \{\{i,\psi(i)\} : i \in M'\} \cup \pi_3.$$

Whence

$$\begin{aligned}\prod_{\Delta\in\pi} \ell_1\{A(\Delta)\} &= \prod_{\Delta\in\pi_1} \ell_1\{A(\Delta)\} \cdot \prod_{i\in M'} \ell_1\{A^i \cap A^{\psi(i)}\} \cdot \prod_{\Delta\in\pi_3} \ell_1\{A(\Delta)\} \\ &= \prod_{\Delta\in\pi_1} \ell_1\{P(\Delta)\} \cdot \prod_{i\in M'} \ell_1\{P^i \cap Q^{\psi(i)-p}\} \cdot \prod_{\Delta\in\pi_2} \ell_1\{Q(\Delta)\}, \end{aligned} \tag{3}$$

where $\pi_2 := \{\Delta - \{p\} : \Delta \in \pi_3\} \in \Pi_N$ where $N := L - \{p\} \subseteq [1,q]$, cf. (1). Clearly, $\pi_1 \in \Pi^p_{\frac{1}{2}(p-2)+k}$ and $\pi_2 \in \Pi^q_k$. Also, $\phi(i) := \psi(i) - p$, defines a permutation ϕ of $[1,q]$. When $k = [q/2]$, M' is either void or has cardinality 1, depending on whether q is even or odd; but the result (3) still holds, on condition that we replace the superscript $\psi(i) - p$ by i in the second factor in case q is odd, and remove this factor itself in case q is even.

On substituting from (3) into (2) and simplifying, we get (a). The result (b) follows as a special case. ■

3.14. *Remarks.* (Inner product formulations) The equality in (1.12) allows us to restate the definition (3.11) of the coefficients $a^p_\pi(P)$ in the form

$$a^n_\pi(P) = \prod_{\Delta\in\pi} (\xi_1(P^{\min\Delta}), \xi_1(P^{\max\Delta}))_{\mathcal{L}_2}.$$

By substituting the corresponding expressions for $a^p_{\pi_1}(P)$, $a^q_{\pi_2}(Q)$ in 3.13, the cross-covariance equality can be stated in terms of such inner products. Such expressions, while cumbersome, are useful in suggesting the formulae which prevail for integrals instead of measures.

Since, cf. (3.4), $\xi_p \in \mathrm{FA}(\mathcal{R}_p, \mathcal{L}_2)$, where $\mathcal{R}_p := \mathrm{ring}(\mathcal{P}_p)$, to show that ξ_p is CA on $\mathcal{R}_p$ we need only show that for any sequence $(R_n)^\infty_{n=1}$ in $\mathcal{R}_p$ such that $R_n \downarrow \emptyset$, we have $|\xi_p(R_n)|_{\mathcal{L}_2} \to 0$. This requires the extension of the covariance theorem 3.13 from $\mathcal{P}_p$ to $\mathcal{R}_p$. (Note that since $\xi_p \in \mathrm{FA}(\mathcal{R}_p, \mathcal{L}_2)$, it makes sense to speak of the covariance $(\xi_p(R), \xi_p(S))_{\mathcal{L}_2}$ for $R, S \in \mathcal{R}_p$.) To get this extension, however, we must confront the fact that whereas the hyperfaces of the intervals P, Q occur in the terms in the expansions in 3.13 and 3.14, the concept of hyperface has no meaning for the sets in $\mathcal{R}_p$, still less for sets in $\mathcal{D}_p$ such as ellipsoids or toroids. Hence before we can extend 3.13 beyond the pre-ring of intervals, *we must reformulate theorem 3.13 in an interval-free fashion, in which hyperfaces are effaced, and only the intervals P, Q and the anatomy of the spaces $\mathbb{R}^p$, $\mathbb{R}^q$ are involved.* This is undertaken in § 4; see (4.20) and 4.21. From here on $(\cdot,\cdot)$ will stand for $(\cdot,\cdot)_{\mathcal{L}_2}$.

4. The diagonal skeletons and the canonical coefficients

A clue as to how the Wiener equality in 3.13 may be freed from dependence on intervals and their hyperfaces may be had by considering the simple case where $p = q = 2$, and P, Q are coordinate rectangles in $\mathbb{R}^2$. Let $I = [x = y]$ be the main diagonal in $\mathbb{R}^2$, ϕ the transposition in $\mathrm{Perm}(2)$, and $\wp_1$ the projection onto the x-axis (or the y-axis), and note that for $P = P^1 \times P^2$,

$$P \cap I = \{(P^1 \cap P^2) \times (P^1 \cap P^2)\} \cap I;$$

whence $\wp_1(P \cap I) = P^1 \cap P^2$. Hence by Wiener's equality 1.18(b),

$$\begin{aligned}(\xi_2(P), \xi_2(Q)) &= \mathbb{E}\{\xi(P^1)\xi(P^2)\xi(Q^1)\xi(Q^2)\} \\ &= \ell_1(P^1 \cap Q^1)\ell_1(P^2 \cap Q^2) + \ell_1(P^1 \cap Q^2)\ell_1(P^2 \cap Q^1) \\ &\quad + \ell_1(P^1 \cap P^2)\ell_1(Q^1 \cap Q^2) \\ &= \ell_2(P \cap Q) + \ell_2(P \cap Q^\phi) + \ell_2[\wp_1(P \cap I) \times \wp_1(Q \cap I)].\end{aligned}$$

Now the RHS of this equality continues to make sense even for non-intervals P, Q in $\mathbb{R}^2$, e.g. for P an ellipse and Q an annulus. This suggests that even for $\mathbb{R}^p$, we may be able to extend the covariance equality 3.13 beyond the intervals, by bringing in the diagonal hyperplanes of $\mathbb{R}^p$.

This section begins with the diagonal anatomy of $\mathbb{R}^p$ and its effects on the measure ℓ_p. It is overwhelmingly combinatorial. The only measures that show up, apart from the Lebesgue, are new non-negative ones concocted to serve combinatorial ends. Vector measure comes in only at the tail end where the nexus with the covariance equality 3.13 is finally established.

Let $p \in \mathbb{N}^+$. The space $\mathbb{R}^p$ has $\binom{p}{2}$ diagonal hyperplanes obtained by setting two coordinates equal. Their union plays a crucial role in the theory. We let

$$(4.1) \qquad \begin{cases} \forall p \geqslant 2, \quad I_{ij}^p := \{x : x \in \mathbb{R}^p \;\&\; x_i = x_j\}, \quad 1 \leqslant i < j \leqslant p; \\ \qquad I_1^p := \bigcup_{i=1}^{p-1} \bigcup_{j=i+1}^{p} I_{ij}^p; \\ \qquad I_1^1 := \emptyset =: I_1^0; \\ \forall p \in \mathbb{N}_+, \quad \mathbb{R}_*^p := \mathbb{R}^p \backslash I_1^p. \end{cases}$$

I_1^p is called the *first 'diagonal skeleton'* of $\mathbb{R}^p$. For $p = 2$, I_1^2 is just the main diagonal of $\mathbb{R}^2$. It is convenient to include the trivial cases $p = 1$ and $p = 0$ in (4.1), as just done.

It is evident that the first diagonal skeletons of $\mathbb{R}^p$, $\mathbb{R}^q$ and $\mathbb{R}^{p+q}$ are related, and that the removal of I_1^p from $\mathbb{R}^p$, results in the dissection of $\mathbb{R}^p$ into a number of disjoint 'half-spaces', akin to the two diagonal half-planes of $\mathbb{R}^2$. The next result gives the precise renditions of these facts:

4.2. Proposition. *Let $p, q \in \mathbb{N}_+$. Then*

(a)

$$I_1^{p+q} = (I_1^p \times \mathbb{R}^q) \cup \bigcup_{i=1}^{p} \bigcup_{j=p+1}^{p+q} I_{ij}^{p+q} \cup (\mathbb{R}^p \times I_1^q),$$

$$I_1^{p+q} = (I_1^q \times \mathbb{R}^p) \cup \bigcup_{i=1}^{q} \bigcup_{j=q+1}^{p+q} I_{ij}^{p+q} \cup (\mathbb{R}^q \times I_1^p),$$

where the Cartesian products are as in 1.30.

(b) *Letting* $\forall \phi \in \operatorname{Perm}(p)$,

$$S_\phi^p := \{t : t = (t_1 \cdots t_p) \in \mathbb{R}^p \;\&\; t_{\phi(1)} < \cdots < t_{\phi(p)}\},$$

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we have

$$\mathbb{R}^p_* := \mathbb{R}^p \backslash I^p_1 = \bigcup_{\phi \in \mathrm{Perm}(p)} S^p_\phi \quad \& \quad S^p_\phi \parallel S^p_\psi, \quad \phi \neq \psi.$$

The proof, which is routine, is omitted. Since by (4.1), $I^1_1 = \emptyset$, the first and second border terms on the RHSs of 4.2(a) drop out according as $p = 1$ or $q = 1$.

Also important for our purposes are the intersections of just those diagonal hyperplanes that arise from the binary-celled partitions of subsets M of $[1, p]$ of even cardinality $2k$, cf. (1.16):

4.3. *Definition.* $\forall p \in \mathbb{N}_+$, $\forall M \subseteq [1, p]$ such that $\#M$ is even, $\& \; \forall \pi \in \Pi_M$,

$$I(\pi, p) := \bigcap_{\Delta \in \pi} I^p_\Delta, \quad \text{where} \quad I^p_\Delta := I^p_{\min \Delta, \max \Delta}, \quad \text{cf. (4.1)}.$$

For $M = \emptyset$ & $\pi \in \Pi_\emptyset = \{\emptyset\}$, cf. (1.16)($a$), we define $I(\pi, p) = I(\emptyset, p) := \mathbb{R}^p$.

It follows readily that

$$(4.4) \quad \begin{cases} (a) \quad \forall p \in \mathbb{N}_+, \quad \forall k \in [0, [p/2]] \;\&\; \forall \pi \in \Pi^p_k, \\ \qquad I(\pi, p) \text{ is a } (p-k)\text{-dimensional subspace of } \mathbb{R}^p; \\ \qquad \text{thus } \dim I(\pi, p) = p - \#\pi, \;\&\text{ when } k \geqslant 1,\; I(\pi, p) \subseteq I^p_1; \\ (b) \quad \forall \text{ even } p \in \mathbb{N}_+ \;\&\; \forall \pi \in \Pi_{[1,p]}, \quad \dim I(\pi, p) = p/2; \\ (c) \quad \forall p, q \in \mathbb{N}_+, \quad \forall \pi \in \bigcup_{k=0}^{[p/2]} \Pi^p_k, \quad I(\pi, p) \times \mathbb{R}^q = I(\pi, p+q), \\ \qquad \text{cf. (1.16)}(d). \end{cases}$$

We shall call the union of all $I(\pi, p)$ for $\pi \in \Pi_M$, as M ranges over all the $2k$ membered subsets of $[1, p]$, *the kth diagonal skeleton of* $\mathbb{R}^p$, in symbols:

$$(4.5) \qquad \forall k \in [0, [p/2]], \qquad I^p_k := \bigcup_{\pi \in \Pi^p_k} I(\pi, p).$$

This yields $I^p_0 := \mathbb{R}^p$, and in particular $I^1_0 = \mathbb{R}$. These higher order skeletons will be encountered as carriers of certain intrinsic measures, cf. 4.15 and 5.2, and will be studied further in § 7.

Also important are the cross-sections of $I(\pi, p)$ obtained by fixing the $p - 2k$ unrestrained coordinates, which we now introduce:

4.6. *Definition.* Let $p \in \mathbb{N}^+$ and $k \in [0, [p/2]]$. Then $\forall \pi \in \Pi^p_k$ and

$$\forall h = (h^1, h^2, \ldots, h^{p-2k}) \in \mathbb{R}^{p-2k},$$

the h-cross-section $I^p_\pi(h)$ of $I(\pi, p)$ is defined by

$$I^p_\pi(h) := I(\pi, p) \cap \wp^{-1}_{M'_\pi}(h),$$

where $M_\pi := \bigcup_{\Delta \in \pi} \Delta$, $M'_\pi := [1, p] \backslash M_\pi$, and the operator $\wp_{M'_\pi}$ is defined as in 1.31.

Thus, if $M'_\pi := [1, p] \backslash M_\pi = \{m_1, m_2, \ldots, m_{p-2k}\}$, where $m_1 < m_2 < \cdots < m_{p-2k}$, we have

$$I^p_\pi(h) := \{x : x \in I(\pi, p) \quad \& \quad x_{m_1} = h^1, x_{m_2} = h^2, \ldots, x_{m_{p-2k}} = h^{p-2k}\}.$$

Note. For $k = 0$ and $\pi \in \Pi_0^p := \{\emptyset\}$, we get

$$\forall h \in \mathbb{R}^p, \qquad I_\pi^p(h) = I_\emptyset^p(h) = \text{the singleton } \{h\} \in \mathbb{R}^p.$$

For p even and $k = [p/2]$, we have $h \in \mathbb{R}^{p-2k} = \mathbb{R}^0 := \{0\}$, $M'_\pi = \emptyset$, and $\pi \in \Pi_{[1,p]}$, and our definition yields $I_\pi^p(0) := I(\pi, p)$.

Example. Let $\pi = \{\{3,4\},\{5,8\},\{6,10\}\} \in \Pi_3^{11}$. Then $\forall h \in \mathbb{R}^5$,

$$\begin{aligned} I_\pi^{11}(h) = \{t : t \in \mathbb{R}^{11} \;\;&\&\;\; t_3 = t_4, t_5 = t_8, t_6 = t_{10}, \\ &\&\;\; t_1 = h^1, t_2 = h^2, t_7 = h^3, t_4 = h^4, t_{11} = h^5\}. \end{aligned}$$

Obviously,

$$(4.7) \qquad \begin{cases} \forall p, k, \pi \;\&\; h, \text{ as in definition 4.6,} \\ \qquad I_\pi^p(h) \text{ is an affine subspace of } \mathbb{R}^p \;\&\; \dim I_\pi^p(h) = k, \\ \qquad \& \displaystyle\bigcup_{h \in \mathbb{R}^{p-2k}} I_\pi^p(h) = I(\pi, p) \subseteq I_k^p, \qquad \text{cf. (4.5).} \end{cases}$$

Let $A \subseteq \mathbb{R}^p$. Then with π as in definition 4.6, $A \cap I(\pi, p)$ is a cross-section of A that lies in the diagonal hyperplane $I(\pi, p)$ of dimension $p - k$. $A \cap I_\pi^p(h)$ is the h cross-section of this cross-section. The dimensionality of this cross-section is k, and accordingly its k-dimensional Lebesgue measure is of interest to us when A is measurable. But the symbol $\ell_k\{A \cap I_\pi^p(h)\}$ makes no sense, since each $t \in A \cap I_\pi^p(h) \subseteq I_\pi^p(h) \subseteq \mathbb{R}^p$ has p components. What we want, strictly speaking, is the $\ell_k(A^*)$, where $A^* \subseteq \mathbb{R}^k$ is the set obtained from $A \cap I_\pi^p(h)$ by eliminating first the $p - 2k$ constant coordinates with values $h^1, h^2, \ldots, h^{p-2k}$, and then from the $2k$-tuple so resulting, further eliminating k superfluous coordinates. This elimination is conveniently affected as follows:

Let $\pi \in \Pi_k^p$ be given by

$$\pi = \{\Delta_1, \Delta_2, \ldots, \Delta_k\} \quad \text{with} \quad \Delta_\alpha = \{i_\alpha, j_\alpha\}, \qquad i_1 < i_2 < \cdots < i_k.$$

Then for $h = (h^1, \ldots, h^{p-2k}) \in \mathbb{R}^{p-2k}$,

$$(4.8) \qquad \begin{cases} A \cap I_\pi^p(h) = \{t : t \in A \;\&\; t_{i_1} = t_{j_1}, \ldots, t_{i_k} = t_{j_k} \\ \qquad\qquad\qquad \& \; t_{m_1} = h^1, \ldots, t_{m_{p-2k}} = h^{p-2k}\}. \end{cases}$$

What interests us is the ℓ_k measure of the 'projection' of $A \cap I_\pi^p(h)$ into $\mathbb{R}^k$, i.e. of the set

$$\{\tau = (t_{i_1}, \ldots, t_{i_k}) : t \in A \cap I_\pi^p(h)\}.$$

Recalling from 1.16(c) that

$$\forall \pi \in \Pi_M, \qquad {}^*\pi := \{\min \Delta : \Delta \in \pi\} \quad \& \quad \pi^* := \{\max \Delta : \Delta \in \pi\},$$

and using the operator $\wp_{\pi^*}$ as defined in 1.31, the 'projection' we want is just $\wp_{\pi^*}\{A \cap I_\pi^p(h)\}$, or equivalently, $\wp_{{}^*\pi}\{A \cap I_\pi^p(h)\}$. That this set is germane to our concerns is clear from the opening paragraph in this section. The next result tells us that for *all intervals* $A \in \mathcal{P}_p$, the $\wp_{\pi^*}\{A \cap I_\pi^p(h)\}$ are the very sets appearing in the covariance equality in 3.13.

4.9. Triviality. *Let* $p \in \mathbb{N}_+$, $k \in [0, [p/2])$, $A \subseteq \mathbb{R}^p$ *and* $P \in \mathcal{P}_p$. *Then* $\forall \pi \in \Pi_k^p \;\&\; M_\pi = \bigcup_{\Delta \in \pi} \Delta \;\&\; \forall h \in \mathbb{R}^{p-2k}$,

(*a*) $\wp_{*\pi}\{A \cap I^p_\pi(h)\} = \wp_{\pi^*}\{A \cap I^p_\pi(h)\}$;
(*b*)

$$\wp_{\pi^*}\{P \cap I^p_\pi(h)\} = \begin{cases} P(\pi), & h \in P_{M'_\pi}, \\ \emptyset, & h \in \mathbb{R}^{p-2k}\backslash P_{M'_\pi}, \quad \text{cf. (3.10)}; \end{cases}$$

(*c*) $\ell_k[\wp_{\pi^*}\{P \cap I^p_\pi(h)\}] = \ell_k\{P(\pi)\}\chi_{P_{M'_\pi}}(h) = a^p_\pi(P)\chi_{P_{M'_\pi}}(h)$, *cf. (3.11)*;
(*d*) *for* $k = 0$, *we have* $\pi = \emptyset$, $h \in \mathbb{R}^p$ & $\ell_0[\wp_{\pi^*}\{P \cap I^p_\pi(h)\}] = \chi_P(h)$;
(*e*) *for even* p *and* $k = p/2$, *we have* $\pi \in \Pi_{[1,p]}$, $h = 0$, &

$$\ell_{p/2}[\wp_{\pi^*}\{P \cap I^p_\pi(0)\}] = \ell_{p/2}\{P(\pi)\}.$$

Proof. (*a*) is clear from the expression (4.8) for $A \cap I^p_\pi(h)$.
(*b*) Let first $k \in [1, [p/2]]$ and

$$M'_\pi = \{m_1, \ldots, m_{p-2k}\}, \qquad m_1 < \cdots < m_{p-2k}.$$

If $\wp_{\pi^*}\{P \cap I^p_\pi(h)\}$ is non-void, so is $P \cap I^p_\pi(h)$, and hence $\exists t$ such that $t \in P \cap I^p_\pi(h)$. Since $t \in I^p_\pi(h)$, therefore $h = (h^1, \ldots, h^{p-2k}) = (t_{m_1}, \ldots, t_{m_{p-2k}})$. But this last vector is in $P_{M'_\pi}$, since $t \in P$. Thus, $h \in P_{M'_\pi}$ whenever $\wp_{\pi^*}\{P \cap I^p_\pi(h)\} \neq \emptyset$.

Next let $h \in P_{M'_\pi}$ and $y \in \wp_{\pi^*}\{P \cap I^p_\pi(h)\}$. Then letting $\pi = \{\Delta_1, \ldots, \Delta_k\}$ with $\Delta_\alpha = \{i_\alpha, j_\alpha\}$, $i_\alpha < j_\alpha$, we have, since $x \in I^p_\pi(h)$,

$$t_{i_\alpha} = t_{j_\alpha} \in P^{i_\alpha} \cap P^{j_\alpha} = P(\Delta_\alpha).$$

Since $\pi^* = \{j_1, \ldots, j_k\}$, it follows that

$$\wp_{\pi^*}(t) = (t_{j_1}, \ldots, t_{j_k}) \in P(\Delta_1) \times \cdots \times P(\Delta_k) =: P(\pi).$$

Thus $\wp_{\pi^*}\{A \cap I^p_\pi(h)\} = P(\pi)$. Thus (*b*) holds for $k \in [1, [p/2]]$.

Next for $k = 0$, we have $\pi = \emptyset$, and cf. the note after 4.6, $I^p_\pi(h) = \{h\}$. Thus

$$P \cap I^p_\pi(h) = P \cap \{h\} = \begin{cases} \{h\} & \text{if } P \cap \{h\} \neq \emptyset, \\ \emptyset & \text{if } P \cap \{h\} = \emptyset. \end{cases}$$

Since $\pi^* = \emptyset$, it follows from (1.33) that

$$\wp_{\pi^*}\{P \cap I^p_\pi(h)\} = \begin{cases} \{0\} & \text{if } P \cap \{h\} \neq \emptyset, \\ \emptyset & \text{if } P \cap \{h\} = \emptyset. \end{cases} \tag{1}$$

Since for $\pi = \emptyset$, $P(\pi) = \{0\}$, cf. (3.10), and moreover $M_\pi = \emptyset$, $M'_\pi = [1, p]$ and $P_{M'_\pi} = P$, (1) can be rewritten:

$$\wp_{\pi^*}\{P \cap I^p_\pi(h)\} = \begin{cases} P(\pi) & \text{for } h \in P_{M'_\pi}, \\ 0 & \text{for } h \in \mathbb{R}^p\backslash P_{M'_\pi}. \end{cases}$$

Thus (*b*) again holds.

(*c*) follows at once from (*b*), and the equality $a^p_\pi(P) := \ell_k\{P(\pi)\}$ in (3.11).

(*d*) For $k = 0$, $\pi = \emptyset$ by (1.16)(*a*), and (1) yields $\ell_0[\wp_{\pi^*}\{P \cap I^p_\pi(h)\}] = \chi_P(h)$.

(*e*) Since for even p, $\Pi^p_{p/2} = \Pi_{[1,p]}$, therefore $\pi \in \Pi_{[1,p]}$ and so $P_{M'_\pi} = P_\emptyset = \{0\}$, cf. 1.35(*a*). From this and part (*b*) it follows that $\wp_{\pi^*}\{P \cap I^p_\pi(h)\} = P(\pi)$, whence (*e*). ■

The LHSs of 4.9(*b*) and 4.9(*c*) will continue to make sense when the interval P is

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replaced by any set $D \subseteq \mathbb{R}^p$, provided that the corresponding set $\wp_{\pi^*}\{D \cap I^p_\pi(h)\}$ falls in the δ-rings $\mathcal{D}_{\#\pi}$, the domain of the Lebesgue measure $\ell_{\#\pi}$. The question arises as to whether this is ensured by the membership of D in $\mathcal{D}_p$. In addressing this and related questions, the next proposition is crucial. It is convenient to fix first a short notation for the complicated sets that we have encountered and shall continue to do:

4.10. *Notation.* $\forall p \in \mathbb{N}_+$, $\forall k \in [0,[p/2]]$, $\forall \pi \in \Pi^p_k$, $\forall h \in \mathbb{R}^{p-2k}$ & $\forall A \subseteq \mathbb{R}^p$,

$$A^p_\pi(h) := \wp_{\pi^*}\{A \cap I^p_\pi(h)\} = \wp_{*\pi}\{A \cap I^p_\pi(h)\}, \qquad \text{cf. } 4.9(a).$$

Note. For $k = 0$, we have by (1.16)(a), $\Pi^p_0 = \{\emptyset\}$, i.e. $\pi = \emptyset = \pi^*$. Also, cf. note to 4.6, $\forall h \in \mathbb{R}^p$, $I^p_\emptyset = \{h\}$. Hence by (1.33), $\forall h \in \mathbb{R}^p$,

$$A^p_\emptyset(h) = p_\emptyset\{A \cap \{h\}\} = \begin{cases} \{0\} & \text{if } h \in A, \\ \emptyset & \text{if } h \in \mathbb{R}^p \backslash A. \end{cases}$$

For even p and $k = [p/2]$, we have (cf. note to 4.6), $\pi \in \Pi_{[1,p]}$, $h \in \mathbb{R}^0 = \{0\}$ and $I^p_\pi(0) = I(\pi, p)$, whence

$$\forall A \subseteq \mathbb{R}^p, \qquad A_\pi(0) = \wp_{\pi^*}\{A \cap I(\pi, p)\}.$$

4.11. Proposition. (Boolean homomorphism) *Let $p \in \mathbb{N}_+$ and $k \in [0,[p/2]]$. Then $\forall \pi \in \Pi^p_k$ & $\forall h \in \mathbb{R}^{q-2k}$, the mapping $A \to A^p_\pi(h)$ is a Boolean homomorphism*[7] *on the algebra of all subsets of $\mathbb{R}^p$ onto the algebra of all subsets of $\mathbb{R}^k$.*

Proof. Let $\pi \in \Pi^p_k$ & $h \in \mathbb{R}^{p-2k}$, and let $\forall \lambda \in \Lambda$, an index set, $A^\lambda \subseteq \mathbb{R}^p$. For simplicity we shall write $A_\pi(h)$ instead of $A^p_\pi(h)$. We then assert that

$$\text{(I)} \qquad \left[\bigcup_{\lambda\in\Lambda} A^\lambda\right]_\pi (h) = \bigcup_{\lambda\in\Lambda} [(A^\lambda)_\pi(h)],$$

$$\text{(II)} \qquad \left[\bigcap_{\lambda\in\Lambda} A^\lambda\right]_\pi (h) = \bigcap_{\lambda\in\Lambda} [(A^\lambda)_\pi(h)].$$

Proof of (I). Since by (4.10), LHS(I) $:= \wp_{\pi^*}[\cup_{\lambda\in\Lambda} A^\lambda) \cap I^p_\pi(h)]$ and RHS(I) $:= \cup_{\lambda\in\Lambda}\wp_{\pi^*}[A^\lambda \cap I^p_\pi(h)]$, the equality follows from simple relation theory.

Proof of (II). For the intersection, rudimentary relation theory yields merely the inclusion:

$$\wp_{\pi^*}\left[\left(\bigcap_{\lambda\in\Lambda} A^\lambda\right) \cap I^p_\pi(h)\right] \subseteq \bigcap_{\lambda\in\Lambda} \wp_{\pi^*}\{A^\lambda \cap I^p_\pi(h)\}.$$

But by virtue of the occurrence of the set $I^p_\pi(h)$, the reverse inclusion also holds. We leave the proof of this to the reader.

Next, $\forall \tau \in \mathbb{R}^k$, we have $\tau = \wp_{\pi^*}(t)$, where t is given by

$$t_{m_1} = h_1, \ldots, t_{m_{p-2k}} = h_{p-2k}; \qquad t_{i_1} = t_{j_1} = \tau_1, \ldots, t_{i_k} = t_{j_k} = \tau_k.$$

Thus $\forall h \in \mathbb{R}^{p-2k}$,

$$\text{(III)} \qquad (\mathbb{R}^p)_\pi(h) = \mathbb{R}^k.$$

[7] That the mapping is not one–one, and therefore not a Boolean isomorphism, is easily seen on taking $p = 2$, $k = 1$, $h = 0$ and $A = P^1 \times P^2$, an interval. Then $A^2_\pi(h) = P^1 \cap P^2$. Thus, for $P^1 \neq P^2$, $(P^1 \times P^2)^2_\pi(h) = (P^2 \times P^1)^2_\pi(h)$, even though $P^1 \times P^2 \neq P^2 \times P^1$.

Finally, for $A' := \mathbb{R}^p \backslash A$, it follows from $\mathbb{R}^p = A \cup A'$, and (III) and (I), that

$$\mathbb{R}^k = (\mathbb{R}^p)_\pi(h) = A_\pi(h) \cup (A')_\pi(h).$$

But by (II), $A_\pi(h) \cap (A')_\pi(h) = (A \cap A')_\pi(h) = \emptyset$. Hence

(IV) $$(A')_\pi(h) = \{A_\pi(h)\}'.$$

By (I)–(IV), we are done. ■

Proposition 4.11 has as a corollary the result we are after:

4.12. Corollary. *Let $p \in \mathbb{N}_+$ and $k \in [0, [p/2]]$. Let the symbol $\mathcal{F}$ stand for one of $\mathcal{P}$, $\mathcal{R}$, $\mathcal{D}$, $\mathcal{B}$. Then*

(a) $E \in \mathcal{F}_p \Longrightarrow \forall \pi \in \Pi_k^p \ \& \ \forall h \in \mathbb{R}^{p-2k}$, $E_\pi^p(h) \in \mathcal{F}_k$; i.e. the mapping $A \to A_\pi^p(h)$ preserves the measurability classes.

(b) $\forall \pi \in \Pi_k^p$, $\forall h \in \mathbb{R}^{p-2k}$ & $\forall E \in \mathcal{D}_k$, $\exists D \in \mathcal{D}_p \ni D_\pi^p(h) = E$; i.e. the mapping $A \to A_\pi^p(h)$ is on $\mathcal{D}_p$ onto $\mathcal{D}_k$.

Proof. (*a*) Let $\pi \in \Pi_k^p$ and $h \in \mathbb{R}^{p-2k}$. First, let $\mathcal{F}$ be $\mathcal{P}$, and let $P \in \mathcal{P}_p$. Then since $P(\pi) := X_{\Delta \in \pi} P(\Delta)$ and $\emptyset$ are in $\mathcal{P}_k$, therefore by 4.9(*b*), $P_\pi^p(h) := \wp_{\pi^*}\{P \cap I_\pi^p(h)\} \in \mathcal{P}_k$.

Next, let $\mathcal{F}$ be $\mathcal{R}$. Since every $R \in \mathcal{R}_p$ is a finite union of sets in $\mathcal{P}_p$, it follows readily from the result for $\mathcal{P}$, and the homomorphic property of the mapping that the implication, $R \in \mathcal{R}_p \implies R_\pi^p(h) \in \mathcal{R}_k$ holds.

Next, let $\mathcal{F}$ be $\mathcal{D}$. Then we have to show that

(I) $$\mathcal{D}_p^* := \{D : D \in \mathcal{D}_p \ \& \ D_\pi^p(h) \in \mathcal{D}_k\} = \mathcal{D}_p.$$

But, as just shown, $\mathcal{R}_p \subseteq \mathcal{D}_p^*$, and cf. [MN, Part III, App. B.5], $\mathcal{D}_p$ is the δ-monotone class generated by $\mathcal{R}_p$. In symbols, we have

$$\mathcal{D}_p = \delta\text{-mon}(\mathcal{R}_p) \subseteq \delta\text{-mon}(\mathcal{D}_p^*) \subseteq \mathcal{D}_p,$$

i.e. $\mathcal{D}_p = \delta\text{-mon}(\mathcal{D}_p^*)$. Hence to prove (I) we need only show that $\mathcal{D}_p^*$ is a δ-monotone class. But this can be easily shown from the implications:

$$\forall n \geqslant 1, \qquad D_n \in \mathcal{D}_p \ \& \ D_n \downarrow D \Longrightarrow (D_n)_\pi^p(h) \downarrow D_\pi^p(h),$$
$$\forall n \geqslant 1, \qquad D_n \in \mathcal{D}_p \ \& \ D_n \uparrow E \in \mathcal{D} \Longrightarrow (D_n)_\pi^p(h) \uparrow D_\pi^p(h),$$

which are clear from the homomorphism of the mapping. Thus (I).

The homomorphism likewise yields the desired implication when $\mathcal{F}$ stands for $\mathcal{B}$. Thus (*a*).

(*b*) Let $M'_\pi := [1,p] \backslash M_\pi = \{m_1, \ldots, m_{p-2k}\}$, $m_1 < \cdots < m_{p-2k}$,

$$\pi = \{\{i_1, j_1\}, \ldots, \{i_k, j_k\}\} \in \Pi_M, \quad \& \quad h \in \mathbb{R}^{p-2k}.$$

Let $E \in \mathcal{D}_k$, and define

$$D := \{t : t \in \mathbb{R}^p \ \& \ (t_{j_1}, \ldots, t_{j_k}) \in E \quad \&$$

(1) $$\forall \alpha \in [1,k], \quad t_{i_\alpha} = t_{j_\alpha} \quad \& \quad \forall \beta \in [1, p-2k], \quad t_\beta = h^{m_\beta}\}.$$

Then obviously

(2) $$D \in \mathcal{B}_p, \quad D \subseteq I_\pi^p(h) \quad \& \quad E = \wp_{\pi^*}(D) = \wp_{\pi^*}\{D \cap I_\pi^p(h)\} =: D_\pi^p(h).$$

Now take $[a, b]$ so large that

$$E \subseteq [a,b]^k \quad \& \quad h \in [a,b]^{p-2k}.$$

Then by (1), $D \subseteq [a,b]^p$ is bounded. Hence by (2),

$$D \cap I^p_\pi(h) = D \in \mathcal{D}_p. \tag{3}$$

By (2) and (3), $E = D^p_\pi(h)$, where $D \in \mathcal{D}_p$. Thus (*b*). ■

Now by 4.9(*c*), the ℓ_k measure of the set $P^p_\pi(h)$, i.e. of $\wp_{\pi^*}\{P \cap I^p_\pi(h)\}$, involves the very coefficients $a^p_\pi(P)$, defined in (3.11), which appear in the covariance equalities in theorem 3.13. This suggest that we standardize the notations for the ℓ_k measure of sets $D^p_\pi(h)$, where $D \in \mathcal{D}_p$. We have the following definition which will play a central role in the rest of this paper:

4.13. *Main definition.* (Canonical coefficients) Let $p \in \mathbb{N}_+$, $k \in [0, [p/2]]$ and $\pi \in \Pi^p_k$. Then $\forall D \in \mathcal{D}_p$ and $\forall h \in \mathbb{R}^{p-2k}$,

$$\lambda^p_\pi(D,h) := \ell_k\{D^p_\pi(h)\}, \qquad \gamma^p_k(D,h) := \sum_{\pi \in \Pi^p_k} \lambda^p_\pi(D,h).$$

Note. For $k = 0$, it readily follows from the note to 4.10 and the definition of ℓ_0 in 3.1 that $\forall D \in \mathcal{D}_p$, and $\pi = \emptyset$,

$$\forall h \in \mathbb{R}^p, \qquad \lambda^p_\emptyset(D,h) = \chi_D(h) = \gamma^p_0(D,h),$$

i.e. $\gamma^p_0(\cdot,h)$ is the unit mass with carrier $\{h\}$.

On taking $k = [p/2]$ we get for even p, $\mathbb{R}^{p-2k} = \mathbb{R}^0 = \{0\}$, and so $\forall D \in \mathcal{D}_p$,

$$\lambda^p_\pi(D,0) = \ell_{p/2}\{D^p_\pi(0)\} = \ell_{p/2}[\wp_{\pi^*}\{D \cap I(\pi,p)\}],$$
$$\gamma^p_{p/2}(D,0) = \sum_{\pi \in \Pi_{[1,p]}} \ell_{p/2}[\wp_{\pi^*}\{D \cap I(\pi,p)\}];$$

and for odd p, $\mathbb{R}^{p-2k} = \mathbb{R}^1$, and so $\forall D \in \mathcal{D}_p$, $\forall \pi \in \Pi^p_{(p-1)/2}$ & $\forall h \in \mathbb{R}^1$,

$$\lambda^p_\pi(D,h) = \ell_{(p-1)/2}\{D^p_\pi(h)\} = \ell_{(p-1)/2}[\wp_{\pi^*}\{D \cap I^p_\pi(h)\}],$$
$$\gamma^p_{(p-1)/2}(D,h) = \sum_{i=1}^{p} \sum_{\pi \in \Pi_{[1,p]\setminus\{i\}}} \ell_{(p-1)/2}[\wp_{\pi^*}\{D \cap I^p_\pi(h)\}].$$

The canonical coefficients are set-functions in the first argument and point functions in the second. We shall now show that the former are CA measures carried by the diagonal skeletons, and that the latter are bounded measurable functions with bounded support. The last is important since, as we shall see, their integrability is crucial. It is worthwhile to first record the ℓ_p-negligibility of the skeleton I^p_1:

$$I^p_1 := \bigcup_{i=1}^{p-1} \bigcup_{j=i+1}^{p} I^p_{i,j} \in \mathcal{N}_{\ell_p}, \quad \text{cf. definition A.2.} \tag{4.14}$$

First, since the diagonal hyperplane $I^p_{i,j}$ is closed, therefore each $I^p_{i,j}$ is a Borel subset of $\mathbb{R}^p$, and so is their finite union I^p_1. Next, since $\dim I^p_{i,j} = p-1$, it is clear that $I^p_{i,j} \in \mathcal{N}_{\ell_p}$, and hence their finite union $I^p_1 \in \mathcal{N}_{\ell_p}$. Thus (4.14).

4.15. Proposition. *Let $p \in \mathbb{N}_+$, $k \in [1,[p/2]]$, and the $\pi \in \Pi^p_k$. Then*

(a) $\forall h \in \mathbb{R}^{p-2k}$, $\lambda^p_\pi(\cdot,h) \in \mathrm{CA}(\mathcal{D}_p, \mathbb{R}_{0+})$, and has $I^p_\pi(h)$, I^p_k and I^p_1 as carriers, and so $\lambda^p_\pi(\cdot,h)$ & $\ell_p(\cdot)$ are mutually singular;

(*b*) $\forall p \in \mathcal{P}_p$ & $\forall h \in \mathbb{R}^{p-2k}$, $\lambda_\pi^p(P,h) = \ell_k\{P(\pi)\}\chi_{P_{M'_\pi}}(h)$;

(*c*) $\forall D \in \mathcal{D}_p$, $\lambda_\pi^p(D,\cdot) \in \mathcal{M}(\mathcal{B}_{p-2k}, B\ell(\mathbb{R}_{0+}))$, *and* $\lambda_\pi^p(D,\cdot)$ *is bounded by* $\ell_1(A)^k$, *and has support* A^{p-2k} *in* $\mathcal{D}_{p-2k}$, *where* $A \in \mathcal{D}_1$ *is such that* $D \subseteq A^p$.

Proof. (*a*) Let $h \in \mathbb{R}^{p-2k}$. Trivially, $\lambda_\pi^p(\cdot,h)$ is FA on $\mathcal{D}_p$. But if $\forall n \geqslant 1$, $D_n \in \mathcal{D}_p$ & $D_n \downarrow \emptyset$, as $n \to \infty$, then obviously $\wp_{\pi^*}\{D_n \cap I_\pi^p(h)\} \downarrow \emptyset$, and hence $\ell_k[\wp_{\pi^*}\{D_n \cap I_\pi^p(h)\}] \downarrow 0$, i.e. $\lambda_\pi^p(D_n,h) \downarrow 0$, as $n \to \infty$. By the Kolmogorov criterion, $\lambda_\pi^p(\cdot,h)$ is CA on $\mathcal{D}_p$. Next from definitions 4.13 and (4.10), namely,

$$\forall h \in \mathbb{R}^{p-2k}, \qquad \lambda_\pi^p(D,h) = \ell_k[\wp_{\pi^*}\{D \cap I_\pi^p(h)\}],$$

it follows that $I_\pi^p(h)$ is a carrier of $\lambda_\pi^p(\cdot,h)$. But by (4.7), (4.4) and (4.14),

$$I_\pi^p(h) \subseteq I(\pi,p) \subseteq I_k^p \cap I_1^p \in \mathcal{N}_{\ell_p}.$$

Thus (*a*).

(*b*) This emerges on combining definition 4.13 and the result 4.9(*c*).

(*c*) Writing $\mathcal{M} := \mathcal{M}(\mathcal{B}_{p-2k}, \mathcal{B}_{[0,\infty)})$, let

$$\mathcal{F} := \{F : F \in D_p \ \&\ \lambda_\pi^p(F,\cdot) \in \mathcal{M}\}.$$

Then as the reader can easily check, $\mathcal{P}_p \subseteq \mathcal{F}$, whence it follows that $\mathcal{D}_p = \delta\text{-mon}(\mathcal{F})$. But since by (*a*), $\lambda_\pi^p(\cdot,h)$ is CA, $\mathcal{F}$ is itself a δ-monotone class, and we have $\mathcal{D}_p = \mathcal{F}$. Thus $\forall D \in \mathcal{D}_p$, $\lambda_\pi^p(D,\cdot) \in \mathcal{M}$.

Finally, $\exists P := A^p \in \mathcal{P}_p$ such that $D \subseteq P$. Thus by (*b*),

$$0 \leqslant \lambda_\pi^p(D,\cdot) \leqslant \ell_k\{P(\pi)\}\chi_{P_{M'_\pi}}(\cdot),$$

i.e. $\lambda_\pi^p(D,\cdot)$ has the bounded support $P_{M'_\pi} = A^{p-2k}$ in $\mathcal{D}_{p-2k}$, cf. (1.35), and is bounded above by $\ell_k\{P(\pi)\} = \ell_1(A^k)$. This establishes (*c*). ∎

As γ_k^p is just a finite sum of λ_π^p, therefore by 4.15(*a*), $\bigcup_{\pi\in\Pi_k^p} I_k^p(h)$ is a carrier of $\gamma_k^p(\cdot,h)$. But, cf. (4.7), I_k^p is a larger set. We thus get the following analogue of 4.15 for γ_k^p:

4.16. Corollary. *Let* $p \in \mathbb{N}_+$ *and* $k \in [1,[p/2]]$. *Then*

(*a*) $\forall h \in \mathbb{R}^{p-2k}$, $\gamma_k^p(\cdot,h) \in \mathrm{CA}(\mathcal{D}_p, \mathbb{R}_{0+})$ *and has the sets* $\bigcup_{\pi\in\Pi_k^p} I_k^p(h)$, I_k^p *and* I_1^p *as carriers, hence* $\gamma_k^p(\cdot,h)$ & $\ell_p(\cdot)$ *are mutually singular;*

(*b*) $\forall D \in \mathcal{D}_p$, $\gamma_k^p(D,\cdot) \in \mathcal{M}(\mathcal{B}_{p-2k}, \mathcal{B}_{[0,\infty)})$, *and* $\gamma_k^p(D,\cdot)$ *is bounded by*

$$\binom{p}{k}\alpha_{2k}\ell_1(A)^k,$$

and has support

$$\bigcup_{\pi\in\Pi_k^p} (A^p)_{M'_\pi} = A^{p-2k} \quad \textit{in} \quad \mathcal{D}_{p-2k},$$

where $A \in \mathcal{D}_1$ *is such that* $D \subseteq A^p$.

We must also attend to the total variations of the measure λ_π^p, γ_k^p, as they play a part in the sequel:

4.17. *Notation.* Let $p \in \mathbb{N}_+$, $k \in [1,[p/2]]$, $\pi \in \Pi_k^p$ & $h \in \mathbb{R}^{p-2k}$, and let

$$\mu_{\pi,h}(\cdot) := \lambda_\pi^p(\cdot,h) \quad \& \quad \nu_{k,h}(\cdot) := \gamma_k^p(\cdot,h) \quad \text{on} \quad \mathcal{D}_p.$$

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Then we shall write

$$|\lambda_\pi^p|(\cdot,h) := |\mu_{\pi,h}|(\cdot) \quad \& \quad |\gamma_k^p|(\cdot,h) := |\nu_{k,h}|(\cdot) \quad \text{on} \quad \mathcal{B}_p.$$

We then have the following extensions of the first and second equalities in definition 4.13 from $\mathcal{D}_p$ to $\mathcal{B}_p$.

4.18. Triviality. *Let p, k, M, π & h be as in 4.17. Then*
(*a*) $\forall A \in \mathcal{B}_p$, $|\lambda_\pi^p|(A,h) = |\ell_k|\{A_\pi^p(h)\}$;
(*b*) $A \in (\mathcal{D}_p)_{\lambda_\pi^p(\cdot,h)} \iff A_\pi^p(h) \in \bar{\mathcal{D}}_k$, *cf.* 1.9;
(*c*)

$$|\gamma_k^p|(\cdot,h) = \sum_{\pi\in\Pi_k^p} |\lambda_\pi^p|(\cdot,h) \quad \text{on} \quad \mathcal{B}_p;$$

(*d*) $\forall A \in \mathcal{B}_p$,

$$|\gamma_k^p|(A,h) = \sum_{\pi\in\Pi_k^p} |\ell_k|[A_\pi^p(h)];$$

(*e*)

$$A \in (\mathcal{D}_p)_{\gamma_k^p(\cdot,h)} \iff \bigcup_{\pi\in\Pi_k^p} A_\pi^p(h) \in \bar{\mathcal{D}}_k.$$

Proof. (*a*), (*b*) The proofs, which are routine, are omitted.
(*c*) This follows at once from (*a*) and the fact that for non-negative measures μ_i, $|\sum_{k=1}^r \mu_i| = \sum_{i=1}^r |\mu_i|$;
(*d*) Substituting for $|\lambda_\pi^p|(A,h)$ from (*a*) into the RHS in (*c*), we get (*d*).
(*e*) We have

$$\begin{aligned} A \in (\mathcal{D}_p)_{|\gamma_k^p|(\cdot,h)} \iff \text{RHS}(b) < \infty &\iff \forall\pi \in \Pi_k^p, \quad |\ell_k|[A_\pi^p(h)] < \infty \\ &\iff \forall\pi \in \Pi_k^p, \quad A_\pi^p(h) \in (\mathcal{D}_k)_{\ell_k} =: \bar{\mathcal{D}}_k. \end{aligned}$$

This last condition is equivalent to that on the RHS(*e*). ∎

Finally we shall show that for sets D in the pre-ring $\mathcal{P}_p$, the $\mathcal{B}_{p-2k}$ measurable functions $\gamma_\pi^p(D,\cdot)$ reduce to simple functions, and that their integration over appropriate subspaces of $\mathbb{R}^p$ yield the expressions on the RHSs of the covariance equality for two intervals, obtained in theorem 3.13.

Combining 4.15(*b*) and 4.9(*c*), we get

$$\lambda_\pi^p(P,h) = a_\pi^p(P)\chi_{P_{M'_\pi}}(h) = \ell_k\{P(\pi)\}\chi_{P_{M'_\pi}}(h), \tag{4.19}$$

and this readily yields

$$\left\{\begin{array}{l} \forall p \in \mathbb{N}_+, \quad \forall k \in [0,[p/2]], \quad \forall\pi \in \Pi_k^p, \quad \forall h \in \mathbb{R}^{p-2k} \;\&\; \forall P \in \mathcal{P}_p, \\ \lambda_\pi^p(P,h) = a_\pi^p(P)\chi_{P_{M'_\pi}}(h) \quad \& \quad \gamma_k^p(P,h) = \displaystyle\sum_{\pi\in\Pi_k^p} a_\pi^p(P)\chi_{P_{M'_\pi}}(h), \\ \gamma_k^p(P,\cdot) = \displaystyle\sum_{\pi\in\Pi_k^p} a_\pi^p(P)\chi_{P_{M'_\pi}}(\cdot) \in S(\mathcal{P}_{p-2k},\mathbb{R}), \\ \text{where } S(\mathcal{F},\mathbb{R}) \text{ is the class of real valued simple functions with cells} \\ \text{in } \mathcal{F}, \text{ cf. 1.1}(c)\text{; in particular, } \gamma_0^p(P,\cdot) = \chi_P(\cdot) = \lambda_\emptyset^p(P,\cdot) \text{ on } \mathbb{R}^p. \end{array}\right. \tag{4.20}$$

Note. The last two equations for the case $k=0$ can be justified as follows. In case $k=0$, by (1.16)(a), $\Pi_0^p = \{\emptyset\}$ and $\pi = \emptyset$. Therefore, $\forall h \in \mathbb{R}^p$,

$$\gamma_0^p(P,h) := \sum_{\pi \in \Pi_0^p} \lambda_\pi^p(P,h) = \lambda_\emptyset^p(P,h).$$

But by (4.9)(d), $\lambda_\emptyset^p(P,h) = \ell_0[\wp_{\emptyset^*}\{P \cap I_\emptyset^p(h)\}] = \chi_P(h)$.

Now let, as in the cross-covariance theorem 3.13, $p,q \in \mathbb{N}_+$ be such that $p+q=2r$ and $q \leqslant p$, $k \in [1,[q/2]]$, $\phi \in \mathrm{Perm}(q-2k)$, and let $M \subseteq [1,p]$, $N \subseteq [1,q]$ be such that $\#M' = \#N' = q-2k$. Then the functions $\gamma^p_{\frac{1}{2}(p-q)+k}(P,\cdot)$, $\gamma_k^q(Q,\cdot)$ being $\mathcal{P}_{q-2k}$-simple, are readily integrable, and it easily follows from (4.20) that

$$\begin{aligned}\int_{\mathbb{R}^{q-2k}} &\gamma^p_{\frac{1}{2}(p-q)+k}(P,h)\cdot\gamma_k^q(Q,h^\phi)\ell_{q-2k}\,(\mathrm{d}h)\\ &= \sum_{\pi_1 \in \Pi^p_{\frac{1}{2}(p-q)+k}} \sum_{\pi_2 \in \Pi_k^q} a^p_{\pi_1}(P) a^q_{\pi_2}(Q)\ell_{q-2k}\{P_{M'_{\pi_1}} \cap (Q_{M'_{\pi_2}})^\phi\}.\end{aligned}$$

The RHS matches the expression for $\Gamma_k^{pq}(P,Q)$ in theorem 3.13, except for the absence of the summation over the class $\mathrm{Perm}(q-2k)$. Thus, upon substituting in 3.13 the expression on the LHS, we get at once the following version of the covariance equality, free of allusion to the hyperfaces of the intervals P, Q:

4.21. Theorem. (Wiener's cross-covariance equality, 'hyperface-free' form) *Let*

(i) $p,q \in \mathbb{N}_+$ *be such that* $p+q=2r$ *is even and* $q \leqslant p$,

(ii) $P \in \mathcal{P}_p$, $Q \in \mathcal{P}_q$.

Then

(a) *when* $p > q$,

$$(\xi_p(P), \xi_q(Q)) = \sum_{k=0}^{[q/2]} \Gamma_k^{pq}(P,Q) \in \mathbb{R}_{0+},$$

where

$$\Gamma_0^{pq}(P,Q) = \sum_{\phi \in \mathrm{Perm}(q)} \int_{\mathbb{R}^q} \gamma^p_{(p-q)/2}(P,h)\chi_Q(h^\phi)\ell_q\,(\mathrm{d}h);$$

for $k \in [1,[q/2]-1]$,

$$\Gamma_k^{pq}(P,Q) = \sum_{\phi \in \mathrm{Perm}(q-2k)} \int_{\mathbb{R}^{q-2k}} \gamma^p_{\frac{1}{2}(p-q)+k}(P,h)\gamma_k^q(Q,h^\phi)\ell_{q-2k}\,(\mathrm{d}h),$$

&

$$\Gamma^{pq}_{[q/2]}(P,Q) = \begin{cases} \gamma^p_{p/2}(P,0)\cdot\gamma^q_{q/2}(Q,0), & q \text{ and } p \text{ even},\\ \displaystyle\int_{\mathbb{R}} \gamma^p_{(p-1)/2}(P,h)\gamma^q_{(q-1)/2}(Q,h)\ell_1\,(\mathrm{d}h), & q \text{ and } p \text{ odd};\end{cases}$$

(b) *when* $p = q$,

$$(\xi_q(P), \xi_q(Q)) = \sum_{k=0}^{[q/2]} \Gamma_k^{qq}(P,Q),$$

where now

$$\varGamma_0^{qq}(P,Q) = \sum_{\phi\in\mathrm{Perm}(q)} \ell_q(P\cap Q^\phi),$$

and for $k\in[1,[q/2]]$, $\varGamma_k^{qq}(P,Q)$ *is given by the same formula as in* (a), *except that now* $p=q$.

Likewise for the expectation of $\xi_p(P)$, in the non-trivial case, p even, we have:

4.22. Proposition. $\forall$ *even* $p\in\mathbb{N}_+$ & $\forall P\in\mathcal{P}_p$, $\mathbb{E}_{\mathbb{P}}\{\xi_p(P)\} = \gamma^p_{p/2}(P,0)$.

Proof. Let p be even and $k=p/2$. Then since $\varPi^p_{p/2} = \varPi_{[1,p]}$, we have $M_\pi = [1,p]$ for any $\pi\in\varPi^p_{p/2}$ and therefore $P_{M'_\pi} = P_\emptyset = \{\emptyset\}$. Thus

$$(1)\qquad \forall\pi\in\varPi^p_{p/2},\qquad \chi_{P_{M'_\pi}}(0) = \chi_{\{0\}}(0) = 1.$$

It follows from (3.12), (1) and (4.20) that

$$\mathbb{E}_{\mathbb{P}}\{\xi_p(P)\} = \sum_{\pi\in\varPi^p_{p/2}} a^p_\pi(P) = \sum_{\pi\in\varPi^p_{p/2}} a^p_\pi(P)\chi_{P_{M'_\pi}}(0) = \gamma^p_{p/2}(P,0).$$

■

5. The countable additivity of ξ_p on the ring $\mathcal{R}_p$ and its extendibility to the δ-ring $\mathcal{D}_p$

We know, cf. (3.4), that $\forall p\in\mathbb{N}_+$, $\xi_p\in\mathrm{FA}(\mathcal{R}_p,\mathcal{L}_2)$. Consequently it makes sense to speak of the covariance $(\xi_p(R),\xi_q(S))$, where $R\in\mathcal{R}_p$, $S\in\mathcal{R}_q$. The question arises as to whether this covariance satisfies the equality given in 4.21. To address this question, we must first define the kernels $\varGamma_k^{pq}$ appearing in 4.21 for sets outside $\mathcal{P}_p$, $\mathcal{P}_q$:

5.1. *Definition.* (The canonical kernels) Let (i) $p,q\in\mathbb{N}_+$ be such that $p+q=2r$ is even and $q\leqslant p$, (ii) $k\in[0,[q/2]]$, (iii) $D\in\mathcal{D}_p$ & $E\in\mathcal{D}_q$. Then

$$\varGamma_k^{pq}(D,E) := \sum_{\phi\in\mathrm{Perm}(q-2k)} \int_{\mathbb{R}^{q-2k}} \gamma^p_{\frac{1}{2}(p-q)+k}(D,h)\gamma^q_k(E,h^\phi)\ell_{q-2k}(\mathrm{d}h).$$

This yields in particular, cf. 4.21:

$$\varGamma_0^{pq}(D,E) := \sum_{\phi\in\mathrm{Perm}(q)} \int_{\mathbb{R}^q} \gamma^p_{(p-q)/2}(D,h)\chi_E(h^\phi)\ell_q(\mathrm{d}h),$$

$$\varGamma_0^{pp}(D,E) := \sum_{\phi\in\mathrm{Perm}(p)} \ell_p(D\cap E^\phi),$$

and

$$\varGamma^{pq}_{[q/2]}(D,E) = \begin{cases} \gamma^p_{p/2}(D,0)\cdot\gamma^q_{q/2}(E,0), & q \text{ and } p \text{ even},\\[2ex] \displaystyle\int_{\mathbb{R}} \gamma^p_{(p-1)/2}(D,h)\gamma^q_{(q-1)/2}(E,h)\ell_1(\mathrm{d}h), & q \text{ and } p \text{ odd}.\end{cases}$$

We shall refer to the $\varGamma_k^{pq}(\cdot,\cdot)$ as the *canonical kernels*. They are all well defined, since by 4.16(b), the integrands are bounded, boundedly supported, and measurable.

These kernels are CA measures in each variable. More precisely we have the following result, which rests in effect on the properties of the canonical coefficients γ_k^p, stated in corollary 4.16.

5.2. Lemma. *Let p, q, r, k be as in 5.1. Then*
(a) $\forall E \in \mathcal{D}_q$, $\Gamma_k^{pq}(\cdot, E) \in \mathrm{CA}(\mathcal{D}_p, \mathbb{R}_{0+})$, *and has* $I^p_{\frac{1}{2}(p-q)+k}$ *as a carrier;*
(b) $\forall D \in \mathcal{D}_p$, $\Gamma_k^{pq}(D, \cdot) \in \mathrm{CA}(\mathcal{D}_q, \mathbb{R}_{0+})$, *and has* I_k^q *as a carrier;*
(c) $\forall D \in \mathcal{D}_p$ *and* $\forall E \in \mathcal{D}_q$,

$$\Gamma_k^{pq}(D, E) = (q-2k)! \int_{\mathbb{R}^{q-2k}} \gamma^p_{\frac{1}{2}(p-q)+k}(D, h) \tilde{\gamma}_k(E, h) \ell_{q-2k}(\mathrm{d}h),$$

where $\tilde{f}$ denotes the symmetrization of the function f.

Proof. (a) Fix $E \in \mathcal{D}_q$ and let $\forall n \in \mathbb{N}_+$, $D_n \in \mathcal{D}_p$ & $D_n \downarrow \emptyset$. Then by 4.16(a),

$$f_n(h) := \gamma^p_{\frac{1}{2}(p-q)+k}(D_n, h)\gamma_k^q(E, h^\phi) \downarrow 0, \quad \text{as} \quad n \to \infty.$$

Next, cf. 4.16(b), $f_n(\cdot)$ is bounded and has bounded support. Hence, by Lebesgue's dominated convergence theorem,

$$\Gamma_k^{pq}(D_n, E) := \sum_{\phi \in \mathrm{Perm}(q-2k)} \int_{\mathbb{R}^{q-2k}} f_n(h) \ell_{q-2k}(\mathrm{d}h) \downarrow 0, \quad \text{as} \quad n \to \infty,$$

i.e. the FA measure $\Gamma_k^{pq}(\cdot, E)$ satisfies the Kolmogorov criterion. Hence $\Gamma_k^{pq}(\cdot, E)$ is CA on $\mathcal{D}_p$.

Finally, $I^p_{\frac{1}{2}(p-q)+k}$ being, by 4.16(a), a carrier of $\gamma^p_{\frac{1}{2}(p-q)+k}(\cdot, h)$ for each h in $\mathbb{R}^{q-2k}$, we see readily from the formula in definition 5.1, that it is also a carrier of $\Gamma_k^{pq}(\cdot, E)$. Thus (a).

(b) is proved similarly.

(c) Take $\sum_{\phi \in \mathrm{Perm}(q-2k)}$ inside the integral in 5.1(iii) and recall that

$$(q-2k)!\tilde{\gamma}_k^q(E, h) = \sum_{\phi \in \mathrm{Perm}(q-2k)} \gamma_k^q(E, h^\phi).$$

■

5.3. *Definition.* Let $p, q \in \mathbb{N}_+$, $q \leqslant p$, and let $\mathcal{F}_p$, $\mathcal{F}_q$ be subfamilies of $\mathcal{D}_p$, $\mathcal{D}_q$ to which the measures ξ_p, ξ_q extend. Then we shall say that *the covariance equality holds for* $\mathcal{F}_p$, $\mathcal{F}_q$ iff $\forall D \in \mathcal{F}_p$ & $\forall E \in \mathcal{F}_q$,

$$(\xi_p(D), \xi_q(E))_{\mathcal{L}_2} = 0, \quad \text{if } p+q \text{ is odd}$$

and, cf. definition 5.1,

$$(\xi_p(D), \xi_q(E))_{\mathcal{L}_2} = \sum_{k=0}^{[q/2]} \Gamma_k^{pq}(D, E) \quad \text{if } p+q \text{ is even.}$$

We now make this affirmation for the rings $\mathcal{R}_p$, $\mathcal{R}_q$:

5.4. Proposition. $\forall p, q \in \mathbb{N}_+$, *the covariance equality holds for the rings* $\mathcal{R}_p$, $\mathcal{R}_q$.

Proof. Let $p, q \in \mathbb{N}_+$, $q \leqslant p$, $R \in \mathcal{R}_p$ and $S \in \mathcal{R}_q$. Then $R = \bigcup_{i=1}^m P_i$, $S = \bigcup_{j=1}^n Q_j$, where $P_i \in \mathcal{P}_p$ are $\|$ and $Q_j \in \mathcal{P}_q$ are $\|$. By definition

$$(1) \qquad \xi_p(R) = \sum_{i=1}^{m} \xi_p(P_i), \qquad \xi_q(S) = \sum_{j=1}^{n} \xi_q(Q_j),$$

whence

$$(2) \qquad (\xi_p(R), \xi_q(S)) = \sum_{i=1}^{m} \sum_{j=1}^{n} (\xi_p(P_i), \xi_q(Q_j)).$$

Now if $p+q$ is odd, then by (3.5) each term on the RHS(2) vanishes and so therefore does the LHS. Next, if $p+q$ is even then by theorem 4.21, for each i, j,

$$(\xi_p(P_i), \xi_q(Q_j)) = \sum_{k=0}^{[q/2]} \Gamma_k^{pq}(P_i, Q_j).$$

Substituting in (2) and changing the order of summation, and using the fact 5.2 that $\Gamma_k^{pq}(\cdot, E)$ and $\Gamma_k^{pq}(D, \cdot)$ are FA, it is easily shown that

$$\Gamma_k^{pq}(R, S) = \Gamma_k^{pq}\left(\bigcup_{i=1}^{m} P_i, \bigcup_{j=1}^{n} Q_j\right) = \sum_{i=0}^{m} \sum_{j=0}^{n} \Gamma_k^{pq}(P_i, Q_j);$$

whence (2) reduces to

$$(\xi_p(R), \xi_q(S)) = \sum_{k=0}^{[q/2]} \Gamma_k^{pq}(R, S).$$

■

To go beyond rings, we need the following:

5.5. Proposition. *$\forall p \in \mathbb{N}_+$, $\xi_p \in \mathrm{CA}(\mathcal{R}_p, \mathcal{L}_2)$ and is locally strongly bounded on $\mathcal{R}_p$, i.e.*

$$\forall p \in \mathbb{N}_+, \quad R_n \in \mathcal{R}_p \cap 2^R, \quad R \in \mathcal{R}_p \ \&\ R_n \text{ are } \| \quad \Longrightarrow \lim_{n\to\infty} \xi_p(R_n) = 0.$$

Proof. An application of the covariance equality in 5.3 with $q = p$ & $S = R = R_n \in \mathcal{R}_p$ & $R_n \subseteq R_1$ yields

$$(1) \qquad |\xi(R_n)|^2 = \sum_{k=0}^{[p/2]} \Gamma_k^{pp}(R_n, R_n) \leqslant \sum_{k=0}^{[p/2]} \Gamma_k^{pp}(R_n, R_1).$$

First let $R_n \downarrow \emptyset$, as $n \to \infty$. Since by lemma 5.2(*a*), each $\Gamma_k^{pp}(\cdot, R_1)$ is CA on $\mathcal{D}_p$, therefore each $\lim_{n\to\infty} \Gamma_k^{pp}(R_n, R_1) = 0$. Next, let $R_n \in \mathcal{R}_p \cap 2^R$, $R \in \mathcal{R}_p$ and the R_n be $\|$. Then each non-negative measure $\Gamma_k^{pp}(\cdot, R_1)$, being CA on $\mathcal{D}_p$ (by 5.2), is certainly locally strongly bounded on $\mathcal{D}_p$, and therefore on $\mathcal{R}_p$. Thus each $\lim_{n\to\infty} \Gamma_k^{pp}(R_n, R_1) = 0$. Hence by (1), on letting $n \to \infty$, we have $|\xi_p(R_n)|^2 \downarrow 0$, in both cases. We thus obtain both the countable additivity of ξ_p on $\mathcal{R}_p$, and its local strong boundedness on $\mathcal{R}_p$. ■

Now thanks to a theorem of Brooks & Dinculeanu (1974) we know that the properties of countable additivity and 'local strong boundedness', affirmed in proposition

5.5 are necessary and sufficient to ensure the existence of a countably additive extension of ξ_p to $\mathcal{D}_p := \delta\text{-ring}(\mathcal{R}_p)$. We accordingly conclude that:

5.6. Theorem. *$\forall p \in \mathbb{N}_+$, the measure ξ_p has an extension to $\mathcal{D}_p$ which is countably additive. Denoting this extension by the symbol ξ_p itself, we have*

$$\xi_p \in \mathrm{CA}(\mathcal{D}_p, \mathcal{L}_2).$$

We show next that the covariance equality extends to δ-rings. The proof rests on the countable additivity of ξ_p:

5.7. Proposition. *$\forall p, q \in \mathbb{N}_+$, the covariance equality holds for $\mathcal{D}_p$, $\mathcal{D}_q$, cf. 5.3.*

Proof. Let first $p+q$ be even. Then by 5.6,

$$\forall Q \in \mathcal{P}_q, \qquad (\xi_p(\cdot), \xi_q(Q)) \in \mathrm{CA}(\mathcal{D}_p, \mathbb{R}),$$

and by 5.2(a)

$$\forall Q \in \mathcal{P}_q, \qquad \sum_{k=0}^{[p/2]} \mathit{\Gamma}_k^{pq}(\cdot, Q) \in \mathrm{CA}(\mathcal{D}_p, \mathbb{R}).$$

These two CA measures on $\mathcal{D}_p$ are, by theorem 4.21(a), equal on $\mathcal{P}_p$. Hence by the identity principle A.8, they are equal on $\mathcal{D}_p$. Thus

$$(1) \qquad \forall D \in \mathcal{D}_p, \qquad (\xi_p(D), \xi_q(Q)) = \sum_{k=0}^{[p/2]} \mathit{\Gamma}_k^{pq}(D, Q).$$

Next, by 5.6,

$$\forall D \in \mathcal{D}_p, \qquad (\xi_p(D), \xi_q(\cdot)) \in \mathrm{CA}(\mathcal{D}_q, \mathbb{R}),$$

and the same argument shows that

$$(2) \qquad \forall D \in \mathcal{D}_p \quad \& \quad \forall E \in \mathcal{D}_q, \qquad (\xi_p(D), \xi_q(E)) = \sum_{k=0}^{[p/2]} \mathit{\Gamma}_k^{pp}(D, E).$$

Next let $p+q$ be odd, $D \in \mathcal{D}_p$ and $E \in \mathcal{D}_q$. It is well known that there exist sequences $(P_n)_1^\infty$ in $\mathcal{P}_p$ and $(Q_n)_1^\infty$ in $\mathcal{P}_q$ such that $\xi_p(P_n) \to \xi_p(D)$ and $\xi_q(Q_n) \to \xi_q(E)$. Thus

$$(3) \qquad (\xi_p(D), \xi_q(E)) = \lim_{n\to\infty} (\xi_p(P_n), \xi_q(Q_n)) = 0, \quad \text{by (3.5).}$$

By (2) and (3) we are done. ■

Next by the Schwartz inequality, $\forall f \in \mathcal{L}_2$, $|\mathbb{E}_\mathbb{P}(f)| \leqslant |f|_{\mathcal{L}_2} |1(\cdot)|_{\mathcal{L}_2} = |f|_{\mathcal{L}_2}$. In particular,

$$(5.8) \qquad \forall D \in \mathcal{D}_p, \qquad 0 \leqslant |\mathbb{E}_\mathbb{P}\{\xi_p(D)\}| \leqslant |\xi_p(D)|_{\mathcal{L}_2}.$$

This, together with the countable-additivity of ξ_p, allows us to extend the results 4.22 and 1.19(b) on the expectation of $\xi_p(P)$ to the δ-ring $\mathcal{D}_p$:

5.9. Proposition. **(The expectation of ξ_p on $\mathcal{D}_p$)** (a) *$\forall$ even $p \in \mathbb{N}_+$ & $\forall D \in \mathcal{D}_p$, we have*

$$\mathbb{E}_\mathbb{P}\{\xi_p(D)\} = \gamma_{p/2}^p(D, 0) = \sum_{\pi \in \mathit{\Pi}_{[1,p]}} \ell_{p/2}\{D_\pi^p(0)\} = \sum_{\pi \in \mathit{\Pi}_{[1,p]}} \ell_{p/2}[\wp_{\pi^*}\{D \cap I(\pi, p\}],$$

&

$$[\mathbb{E}_{\mathbb{P}}\{\xi_p(D)\}]^2 = \Gamma^{pp}_{p/2}(D, D).$$

(*b*) $\forall$ *odd* $p \in \mathbb{N}_+$ & $\forall D \in \mathcal{D}_p$, $\mathbb{E}_{\mathbb{P}}\{\xi_p(D)\} = 0$.

Proof. (*a*) To prove the first equality, note that since $\mathbb{E}_{\mathbb{P}} \in \mathcal{L}_2'$, therefore

$$\mathbb{E}_{\mathbb{P}}\{\xi_p(\cdot)\} \in \mathrm{CA}(\mathcal{D}_p, \mathbb{R}).$$

Since by 4.16(*a*), $\gamma^p_{p/2}(\cdot, 0) \in \mathrm{CA}(\mathcal{D}_p, \mathbb{R})$, and by 4.22, the two measures are equal on $\mathcal{P}_p$, therefore by the identity principle A.8, they are equal on $\mathcal{D}_p$. This proves the equality on the left. The others follow from definitions 4.13 and (4.10).

Next, combining definition 5.1 (et seq.) and the last equality, we get

$$\Gamma^{pp}_{p/2}(D, D) = \gamma^p_{p/2}(D, 0)\gamma^p_{p/2}(D, 0) = [\mathbb{E}\{\xi_p(D)\}]^2.$$

This completes the proof of (*a*).

(*b*) This is an easy consequence of the result (3.9), since each

$$\xi_p(D) = \lim_{n\to\infty} \xi_p(P_n)$$

for suitable $P_n \in \mathcal{P}_p$. ■

We turn next to the factorization of the measure $\xi_{p+q}(D \times E)$, where $D \in \mathcal{D}_p$, $E \in \mathcal{D}_q$. Ancillary to this factorization is the classical factorization of the δ-ring $\mathcal{D}_{p+q}$ itself:

(5.10) $$\forall p, q \in \mathbb{N}_+, \qquad \mathcal{D}_{p+q} = \delta\text{-ring}(\mathcal{D}_p \times \mathcal{D}_q).$$

The proof that $\xi_{p+q} = \xi_p \times \xi_q$ is not obvious, since $|\cdot|_{\mathcal{L}_2}$ is not a Banach algebra norm, and simple inferences such as $\xi(D_n) \to 0$ in $\mathcal{L}_2$, implies that $\xi(D_n)\eta(E) \to 0$ in $\mathcal{L}_2$, where $E \in \mathcal{D}_q$ is fixed, fail. However, for the specific measures ξ_p, ξ_q, this implication does hold, and the proof can be completed. But it involves several steps.

5.11. Theorem. (Product measure) *Let $p, q \in \mathbb{N}_+$. Then $\xi_{p+q} = \xi_p \times \xi_q$, i.e. we have*

$$\forall D \in \mathcal{D}_p \quad \& \quad \forall E \in \mathcal{D}_q, \qquad D \times E \in \mathcal{D}_{p+q} \quad \& \quad \xi_{p+q}(D \times E) = \xi_p(D) \cdot \xi_q(E).$$

Proof. We shall show in succession that:

(I) $\forall R \in \mathcal{R}_p$ & $\forall S \in \mathcal{R}_q$, $\quad \xi_p(R)\xi_q(S) = (\xi_p \times \xi_q)(R \times S)$;

(II) $\forall R \in \mathcal{R}_q$, $\quad \xi_p(R)\xi_q(\cdot) \in \mathrm{CA}(\mathcal{D}_q, \mathcal{L}_2)$;

(III) $\forall R \in \mathcal{R}_p$ & $\forall E \in \mathcal{D}_q$, $\quad \xi_p(R)\xi_q(E) = (\xi_p \times \xi_q)(R \times E)$;

(IV) $\forall E \in \mathcal{D}_q$, $\quad \xi_p(\cdot)\xi_q(E) \in \mathrm{CA}(\mathcal{D}_p, \mathcal{L}_2)$;

(V) $\forall D \in \mathcal{D}_p$ & $\forall E \in \mathcal{D}_q$, $\quad \xi_p(D)\xi_q(E) = (\xi_p \times \xi_q)(D \times E)$.

Proof of (I). Let $P = P^1 \times \cdots \times P^p \in \mathcal{P}_p$ and $Q = Q^1 \times \cdots \times Q^q \in \mathcal{P}_q$. Then it follows from the definition (1.13) of ξ_p, ξ_q, ξ_{p+q} that

$$\xi_{p+q}(P \times Q) = \xi(P^1) \cdots \xi(P^p) \cdot \xi(Q^1) \cdots \xi(Q^q) = \xi_p(P) \cdot \xi_q(Q).$$

Thus

(1) $$\forall P \in \mathcal{P}_p \quad \& \quad \forall Q \in \mathcal{P}_q, \qquad \xi_{p+q}(P \times Q) =: \xi_p(P) \cdot \xi_q(Q).$$

Since sets in $\mathcal{R}_p$ and $\mathcal{R}_q$ are finite disjoint unions of intervals in $\mathcal{P}_p$ and $\mathcal{P}_q$, and ξ_p, ξ_q, ξ_{p+q} are FA, therefore (I) easily follows from (1).

Proof of (II). Let $R \in \mathcal{R}_p$. Then by (I),

$$\xi_p(R)\xi_q(\cdot) = \xi_{p+q}(R \times \cdot) \quad \text{on} \quad \mathcal{R}_q. \tag{2}$$

But by theorem 5.6, ξ_{p+q} is CA on $\mathcal{D}_{p+q}$, i.e. by (5.10) on δ-ring$(\mathcal{D}_p \times \mathcal{D}_q)$. Therefore, obviously $\xi_{p+q}(R \times \cdot)$ is CA on $\mathcal{D}_q$. It is therefore also locally strongly bounded on $\mathcal{D}_q$, and therefore on $\mathcal{R}_q$. Thus by (2), $\xi_p(R)\xi_q(\cdot)$ is CA and locally strongly bounded on $\mathcal{R}_q$. But this, by the Brooks & Dinculeanu theorem (cited on p. 1138), yields (II).

Proof of (III). Let $R \in \mathcal{R}_p$. Since ξ_{p+q} is CA on $\mathcal{D}_{p+q}$ and therefore on $\mathcal{D}_p \times \mathcal{D}_q$, we have $\xi_{p+q}(R \times \cdot) \in \mathrm{CA}(\mathcal{D}_q, \mathcal{L}_2)$. And by (II), $\xi_p(R)\xi_q(\cdot) \in \mathrm{CA}(\mathcal{D}_q, \mathcal{L}_2)$. Also by (I), these measures are equal to $\mathcal{R}_q$. Hence by the identity principle A.8, they are equal on δ-ring$(\mathcal{R}_q)$, i.e. on $\mathcal{D}_q$. Thus (III).

Proof of (IV). Let $E \in \mathcal{D}_q$. Then by (III),

$$\xi_p(\cdot)\xi_q(E) = \xi_{p+q}(\cdot \times E) \quad \text{on} \quad \mathcal{R}_p. \tag{3}$$

Then as in the proof of (II), it follows that $\xi_p(\cdot)\xi_q(E)$ is both CA and locally strongly bounded on $\mathcal{R}_p$, whence (IV) follows from the Brooks & Dinculeanu theorem (1974).

Proof of (V). Let $E \in \mathcal{D}_q$. Then by 5.6 and (IV), both $\xi_{p+q}(\cdot \times E)$ and $\xi_p(\cdot)\xi_q(E)$ are CA on $\mathcal{D}_p$, and by (III) they are equal on $\mathcal{R}_p$. Hence by the identity principle A.8, we have (V). ■

The result 5.11 obviously extends to any finite number of factors, and therefore $\{\xi_p(D)(\omega)\}^r = \xi_{pr}(D^r)(\omega)$, $\forall D \in \mathcal{D}_p$. An easy corollary of this equality and (5.8) is the following generalization of 3.3:

5.12. Corollary. *Let $p \in \mathbb{N}_+$. Then $\forall D \in \mathcal{D}_p$ & $\forall r \in \mathbb{N}_+$, $\xi_p(D) \in \mathcal{L}_r$, and*

$$|\xi_p(D)|^r_{\mathcal{L}_r} := \mathbb{E}_{\mathbb{P}}\{|\xi_p(D)|^r\} \leqslant |\xi_{pr}(D^r)|_{\mathcal{L}_2} < \infty.$$

Thus the random variable $\xi_p(D)$ has finite (raw absolute) moments of all orders.

It is quite easily seen that the measures ξ_p are robust in the sense that their total variations $|\xi_p|$ have the binary range $\{0, \infty\}$; specifically:

$$\forall p \in \mathbb{N}_+, \qquad D \in \mathcal{D}_p \quad \& \quad \xi_p(D) \neq 0 \Longrightarrow |\xi_p|(D) = \infty.$$

The appropriate variation for the ξ_p are the quasi- and semi-variations q_{ξ_p} and s_{ξ_p}, defined in A.3. Even though the ξ_p are not CAOS, they share with the CAOS measures the property that q_{ξ_p} and s_{ξ_p} are equal to the norm of ξ_p. More precisely, we have:

5.13. Proposition. (Quasi- and semi-variations) *Let $p \in \mathbb{N}_+$. Then*
(*a*) $\forall D \in \mathcal{D}_p$, $s_{\xi_p}(D) = |\xi_p(D)| = q_{\xi_p}(D) \in \mathbb{R}_{0+}$;
(*b*) $\forall A \in \mathcal{B}_p$, $s_{\xi_p}(A) = q_{\xi_p}(A) \in [0, \infty]$;
(*c*) $\forall D, E \in \mathcal{D}_p$, $D \subseteq E \Longrightarrow |\xi_p(D)| \leqslant |\xi_p(E)|$.

Proof. (*a*) Let $D \in \mathcal{D}_p$. Then by (A.4),

$$s_{\xi_p}(D) = \sup_{\substack{|x'| \leqslant 1 \\ x' \in (\mathcal{L}_2)'}} |x' \circ \xi_p|(D),$$

$(\mathcal{L}_2)'$ being the dual of $\mathcal{L}_2$. It follows readily that

$$|\xi_p(D)| \leqslant q_{\xi_p}(D) \leqslant s_{\xi_p}(D). \tag{1}$$

To show the reverse inequality, let

$$\mathit{\Pi}_D := \{\pi : \pi \text{ is a finite class of } \| \text{ sets in } \mathcal{D}_p \cap 2^D\}.$$

Let $\pi = \{\Delta_1, \Delta_2, \ldots, \Delta_n\} \in \Pi_D$ & $\alpha(\cdot) \in \mathbb{R}^\pi$ be such that $\forall i \in [1, n]$, $|\alpha(\Delta_i)| \leqslant 1$. Then by the covariance equality (cf. 5.3 and 5.7),

$$\left|\sum_{\Delta\in\pi} \alpha(\Delta)\xi_p(\Delta)\right|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha(\Delta_i)\alpha(\Delta_j)(\xi_p(\Delta_i), \xi_p(\Delta_j))$$

$$(2) \qquad = \sum_{k=0}^{[p/2]}\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha(\Delta_i)\alpha(\Delta_j)\cdot \Gamma_k^{pp}(\Delta_i, \Delta_j).$$

Now it is easy to see that

$$\mathrm{RHS}(2) \leqslant \sum_{k=0}^{[p/2]}\sum_{i=1}^{n}\sum_{j=1}^{n} \Gamma_k^{pp}(\Delta_i, \Delta_j).$$

Also, from the finite additivity of the $\Gamma_k^{pp}(\cdot,\cdot)$ in each variable, we get with $S := \bigcup_{i=1}^{p} \Delta_i$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \Gamma_k^{pp}(\Delta_i, \Delta_j) = \Gamma_k^{pp}(S, S) \leqslant \Gamma_k^{pp}(D, D).$$

We can thus conclude from (2) and 5.7 that

$$(3) \qquad \left|\sum_{\Delta\in\pi} \alpha(\Delta)\xi_p(\Delta)\right|^2 \leqslant \sum_{k=0}^{[p/2]} \Gamma_k^{pp}(D, D) = |\xi_p(D)|^2.$$

On taking the square root and then the suprema over $\alpha(\cdot)$ and over π on the LHS, we get $s_{\xi_p}(D) \leqslant |\xi_p(D)|$. This, together with (1), yields the equalities in (*a*).

(*b*) Let $A \in \mathcal{B}_p$. Then since $\mathcal{B}_p = \mathcal{D}_p^{\mathrm{loc}}$, cf. §1, it follows from (*a*), $\forall D \in \mathcal{D}_p$, $q_{\xi_p}(D \cap A) = s_{\xi_p}(D \cap A)$. Taking the supremum over $D \in \mathcal{D}_p$, we get (*b*), cf. A.3(*b*).

(*c*) follows at once from (*a*) and the monotonicity of the quasi-variation. ∎

We turn next to the Lebesgue decomposition of ξ_p with respect to ℓ_p. First, recall that since $\mathcal{D}_p^{\mathrm{loc}} = \mathcal{B}_p$, a set A is ξ_p-*negligible* in the sense of definition A.2, in symbols $A \in \mathcal{N}_{\xi_p}$, iff

$$A \in \mathcal{B}_p \quad \& \quad \forall D \in \mathcal{D}_p, \qquad \xi(D \cap A) = 0;$$

and C is a *carrier of* ξ_p in the sense of definition A.2, iff

$$C \in \mathcal{B}_p \quad \& \quad \forall D \in \mathcal{D}_p, \qquad \xi_p(D) = \xi_p(D \cap C).$$

We say that two vector-valued FA measures ξ, η on the same δ-ring are mutually singular, in symbols $\eta \perp\!\!\!\perp \xi_p$, iff they possess carriers that are $\|$. The notion of *absolute continuity* of a vector-valued measure ξ with respect to a non-negative measure μ, in symbols $\xi \lll \mu$, is defined in [MN, I, def. 2.36]. From [MN, II, prop. 3.6], we know that

$$(5.14) \quad \forall \rho \in \mathrm{CA}(\mathcal{D}_p, \mathcal{L}_2), \quad \rho \lll \ell_p \quad \text{iff} \quad D \in \mathcal{D}_p \quad \& \quad \ell_p(D) = 0 \Longrightarrow \rho(D) = 0.$$

By (A.5), this result can be rendered in the form,

$$(5.15) \qquad \rho \lll \ell_p \quad \text{iff} \quad B \in \mathcal{B}_p \quad \& \quad |\ell_p|(B) = 0 \Longrightarrow s_\rho(B) = 0.$$

Now define, cf. (4.1):

$$(5.16) \quad \forall p \in \mathbb{N}_+ \quad \& \quad \forall D \in \mathcal{D}_p, \quad \xi_p^a(D) := \xi_p(D \backslash I_1^p), \qquad \xi_p^b(D) := \xi_p(D \cap I_1^p).$$

We can then assert the following lemma:

5.17. Lemma. *Let $p \in \mathbb{N}_+$. Then*
(*a*) $\xi_p^a, \xi_p^b \in \mathrm{CA}(\mathcal{D}_p, \mathcal{L}_2)$, $\mathbb{E}_{\mathbb{P}}\{\xi_p^a(\cdot)\} = 0$ *on* $\mathcal{D}_p$;
(*b*) $\forall D, E \in \mathcal{D}_p$,

$$(\xi_p^a(D), \xi_p^a(E)) = \sum_{\phi \in \mathrm{Perm}(p)} \ell_p(D \cap E^\phi);$$

(*c*) $\forall D \in \mathcal{D}_p,\ \sqrt{\ell_p(D)} \leqslant q_{\xi_p^a}(D) = |\xi_p^a(D)| = s_{\xi_p^a}(D) \leqslant \sqrt{p!}\sqrt{\ell_p(D)}$;
(*d*) $\forall A \in \mathcal{B}_p,\ \sqrt{|\ell_p|(A)} \leqslant q_{\xi_p^a}(A) = s_{\xi_p^a}(A) \leqslant \sqrt{p!}\sqrt{|\ell_p|(A)}$.

Proof. (*a*) The first statement is obvious from (5.16) and theorem 5.6. Next let $D \in \mathcal{D}_p$. Then by (5.16) and 5.9,

$$\mathbb{E}_{\mathbb{P}}\{\xi_p^a(D)\} = \mathbb{E}_{\mathbb{P}}\{\xi_p^a(D \backslash I_1^p)\} = \gamma_{p/2}^p(D \backslash I_1^p, 0) = 0,$$

since by 4.16(*a*), $\gamma_{p/2}^p(\cdot, 0)$ has carrier $I_{p/2}^p \subseteq I_1^p$. Thus (*a*).

(*b*) Let $D, E \in \mathcal{D}_p$. Then by the covariance equality,

$$(1) \qquad (\xi_p^a(D), \xi_p^a(E)) = (\xi_p(D \backslash I_1^p), \xi_p(E \backslash I_1^p)) = \sum_{k=0}^{[p/2]} \Gamma_k^{pp}(D \backslash I_1^p, E \backslash I_1^p).$$

But since by lemma 5.2(*a*), $\Gamma_k^{pp}(\cdot, E)$ has the carrier I_k^p, and obviously $I_k^p \subseteq I_1^p$ for $k \geqslant 1$, it follows that the terms $\Gamma_k^{pp}(D \backslash I_1^p, E \backslash I_1^p) = 0$, for $1 \leqslant k \leqslant [p/2]$. Thus from (1),

$$(\xi_p^a(D), \xi_p^a(E)) = \Gamma_0^{pp}(D \backslash I_1^p, E \backslash I_i^p) = \sum_{\phi \in \mathrm{Perm}(p)} \ell_p[(D \backslash I_1^p) \cap (E \backslash I_1^p)^\phi],$$

by 5.1. But since, cf. (4.14), $I_i^p \in \mathcal{N}_{\ell_p}$, the RHS reduces to the RHS(*b*).

(*c*) Let $D \in \mathcal{D}_p$. The two equalities follow at once on applying those in proposition 5.13(*a*) to the set $D \backslash I_1^p$. It remains to show

$$(\mathrm{I}) \qquad \sqrt{\ell_p(D)} \leqslant |\xi_p^a(D)| \leqslant \sqrt{p!}\sqrt{\ell_p(D)}.$$

Proof of (I). By (*b*) we have

$$|\xi_p^a(D)|^2 = \sum_{\phi \in \mathrm{Perm}(p)} \ell_p(D \cap D^\phi).$$

Since the RHS obviously exceeds the term with $\phi = I$, we get the first inequality in (I). Next, since for each ϕ, $\ell_p(D \cap D^\phi) \leqslant \ell_p(D)$, we get the second inequality in (I). Thus (I) holds, and (*c*) is proved.

(*d*) follows on applying (*c*) to the sets $A \cap D$, and taking the supremum over $D \in \mathcal{D}_p$. ■

5.18. Theorem. (Lebesgue decomposition) *Let $p \in \mathbb{N}_+$. Then with ξ_p^a, ξ_p^b as in (5.16),*
(*a*) $\xi_p = \xi_p^a + \xi_p^b,\ \xi_p^a \ll \ell_p \perp\!\!\!\perp \xi_p^b$;
(*b*) ξ_p^a *and* ℓ_p *are equivalent, i.e.* $\mathcal{N}_{\xi_p^a} = \mathcal{N}_{\ell_p}$.

Proof. (*a*) The equality is obvious from (5.16). Next from lemma 5.17(*c*) and (5.14), we see that $\xi_p^a \ll \ell_p$. Finally, by (4.14), $\mathbb{R}^p \backslash I_1^p$ is a carrier of ℓ_p and by

(5.16), I_1^p is a carrier of ξ_p^b. Thus $\ell_p \perp\!\!\!\perp \xi_p^b$. This establishes (a). (b) is clear from lemma 5.17(c). ■

The following orthogonality properties of the Lebesgue components of ξ_p are immediate from the disjointness of their carriers, cf. (5.16), and the definitions 4.13 and 5.1:

5.19. Corollary. (a) *The ranges of* ξ_p^a, ξ_p^b *are orthogonal:*

$$\forall D, E \in \mathcal{D}_p, \qquad \xi_p^a(D) \perp \xi_p^b(E);$$

(b)

$$\forall D \in \mathcal{D}_p, \qquad |\xi_p(D)|^2 = |\xi_p^a(D)|^2 + |\xi_p^b(D)|^2.$$

It follows from 5.18(a), 5.19(a) that

$$(\xi_p(D), \xi_p(E)) = (\xi_p^a(D), \xi_p^a(E)) + (\xi_p^b(D), \xi_p^b(E)).$$

But by 5.17(b), and the equality for $\Gamma_0^{pp}(D,E)$ in 5.1, we see that

$$(5.20) \qquad (\xi_p^a(D), \xi_p^a(E)) = \Gamma_0^{pp}(D, E).$$

It follows from the last two equalities, and the covariance equality in 5.3 with $q = p$ that

$$(5.21) \qquad \forall D, E \in \mathcal{D}_p, \qquad (\xi_p^b(D), \xi_p^b(E)) = \sum_{k=1}^{[p/2]} \Gamma_k^{pp}(D, E).$$

The semi-variation measure of the Lebesgue components are given in the next triviality. The proof, which appeals to the classical triviality that if μ_0 is defined on $\mathcal{D}$ by $\mu_0(\cdot) = \mu(\cdot \cap B)$, where $B \in \mathcal{D}^{\mathrm{loc}}$, then $|\mu_0|(\cdot) = |\mu|(\cdot \cap B)$ on $\mathcal{D}^{\mathrm{loc}}$, is obvious:

5.22. Triviality. (a) $\forall x' \in \varepsilon(\mathcal{L}_2)'$ & $\forall A \in \mathcal{B}_p$,

$$|x' \circ \xi_p^a|(A) = |x' \circ \xi_p|(A \backslash I_1^p), \qquad |x' \circ \xi_p^b|(A) = |x' \circ \xi_p|(A \cap I_1^p),$$
$$|x' \circ \xi_p|(A) = |x' \circ \xi_p^a|(A) + |x' \circ \xi_p^b|(A);$$

(b) $\forall A \in \mathcal{B}_p$, $s_{\xi_p^a}(A) = s_{\xi_p}(A \backslash I_1^p)$ & $s_{\xi_p^b}(A) = s_{\xi_p}(A \cap I_1^p)$.

As for a *control measure* for the measure ξ_p itself, cf. Traynor (1973), we can show, as the reader can easily check, that the covariance equality yields an explicit formulae for $|\xi_p(D)|^2$, and this suggests a natural control measure μ_p for ξ_p. More precisely:

5.23. Corollary. (Control measure for ξ_p) *Let* $p \in \mathbb{N}_+$. *Then, with* ˜ *indicating symmetrization,*

(a) $\forall D \in \mathcal{D}_p$,

$$|\xi_p(D)|^2 = p! \sum_{k=0}^{[p/2]} \int_{\mathbb{R}^{p-2k}} |\tilde{\gamma}_k^p(D, h)|^2 \ell_{p-2k}\,(\mathrm{d}h).$$

(b) *Defining* μ_p *on* $\mathcal{D}_p$ *by*

$$\forall D \in \mathcal{D}_p, \qquad \mu_p(D) := \sum_{k=0}^{[p/2]} \int_{\mathbb{R}^{p-2k}} \tilde{\gamma}_k^p(D, h) \ell_{p-2k}\,(\mathrm{d}h),$$

we have $\mu_p \in \mathrm{CA}(\mathcal{D}_p, \mathbb{R}_{0+})$, *and* ξ_p *is locally Lipchitzian with respect to* μ_p, *i.e.* $\forall A \in \mathcal{P}_1$, $\exists$ *a constant* β_A *such that*

$$\forall D \in \mathcal{D}^p \cap 2^{A^p}, \qquad |\xi_p(D)|^2 \leqslant \beta_A \cdot \mu_p(D).$$

The measures ξ_p *and* μ_p *are equivalent, i.e.* $\mathcal{N}_{\xi_p} = \mathcal{N}_{\mu_p}$.

5.24. *Remarks on product measures.* Let $\mathcal{C}$, $\mathcal{D}$ be δ-rings over sets S, T, respectively, and let

$$\rho \in \mathrm{CA}(\mathcal{C}, \mathcal{L}_2), \qquad \sigma \in \mathrm{CA}(\mathcal{D}, \mathcal{L}_2). \tag{1}$$

We define $\rho \times \sigma$ on the pre-ring $\mathcal{C} \times \mathcal{D}$ by

$$\forall C \times D \in \mathcal{C} \times \mathcal{D}, \qquad (\rho \times \sigma)(C \times D) := \rho(C) \cdot \sigma(D). \tag{2}$$

In general, $\rho(C) \cdot \sigma(D) \notin \mathcal{L}_2$. However, by the Schwartz inequality,

$$|(\rho \times \sigma)(C \times E)|_{\mathcal{L}_2} \leqslant |\rho(C)|_{\mathcal{L}_4} \cdot |\sigma(D)|_{\mathcal{L}_4}.$$

Assume that $\rho(C)$ and $\sigma(D)$ have finite 4th moments, $\forall C \in \mathcal{C}$ and $\forall D \in \mathcal{D}$. Then it easily follows that

$$\rho \times \sigma \in \mathrm{FA}(\mathcal{C} \times \mathcal{D}, \mathcal{L}_2). \tag{3}$$

But, unlike the classical situation, we cannot in general conclude that $\rho \times \sigma \in \mathrm{CA}(\mathcal{C} \times \mathcal{D}, \mathcal{L}_2)$.

In this regard, the special case $\rho = \xi_p$, $\sigma = \xi_q$ is a notable exception, for by theorems 5.11 and 5.6,

$$\forall p, q \in \mathbb{N}_+, \qquad \xi_p \times \xi_q = \xi_{p+q} \in \mathrm{CA}(\mathcal{D}_{p+q}, \mathcal{L}_2). \tag{5.25}$$

We shall now deduce from (5.25) the countable additivity of the products of the Lebesgue components of ξ_p. We assert:

5.26. Proposition. *Let* $p, q \in \mathbb{N}_+$. *Then*

$$\xi_p^a \times \xi_q^a, \quad \xi_p^a \times \xi_q^b, \quad \xi_p^b \times \xi_p^a, \quad \xi_p^b \times \xi_q^b \quad \in \quad \mathrm{CA}(\mathcal{D}_{p+q}, \mathcal{L}_2).$$

Proof. Let $\mathcal{R} := \mathrm{ring}(\mathcal{D}_p \times \mathcal{D}_q)$. It will suffice to deal with the product $\rho := \xi_p^a \times \xi_q^a$. Trivially, cf. 5.24(3), we have

$$\rho \in \mathrm{FA}(\mathcal{R}, \mathcal{L}_2).$$

Now grant momentarily that

$$|\rho(\cdot)| \leqslant |\xi_{p+q}(\cdot)| \quad \text{on} \quad \mathcal{R}. \tag{I}$$

Then since by 5.14, $\xi_{p+q} \in \mathrm{CA}(\mathcal{D}_{p+q}, \mathcal{L}_2)$, it satisfies the Kolmogorov condition on $\mathcal{R}$, i.e. $R_n \in \mathcal{R}$ and $R_n \downarrow \emptyset \Longrightarrow \xi_{p+q}(R_n) \to 0$. It follows from (I) that so does $\rho(\cdot)$. Hence $\rho(\cdot)$ is CA on $\mathcal{R}$. Next, ξ_{p+q} being CA is locally strongly bounded on $\mathcal{D}_{p+q}$ and therefore on $\mathcal{R}$, cf. 5.5 It follows from (I) that so is $\rho(\cdot)$. Hence by the Brooks–Dinculeanu theorem (1974), ρ has a CA extension to δ-ring$(\mathcal{R})$, i.e. cf. (5.10) to $\mathcal{D}_{p+q}$. Thus it only remains to prove (I).

Proof of (I). Let $R \in \mathcal{R}$. Then

$$R = \bigcup_{k=1}^{r} (D_k \times E_k), \qquad (D_k \times E_k) \in \mathcal{D}_p \times \mathcal{D}_q \text{ and are } \|.$$

Appealing to the definitions (5.16) of ξ_p^a, ξ_p^b and the equality (5.25), we get

$$\rho(R) = \xi_{p+q}\left[\bigcup_{k=1}^{r}\{(D_k\backslash I_1^p)\times(E_k\backslash I_1^q)\}\right].$$

Hence by the monotonicity stated in 5.13(c),

$$(1)\qquad |\rho(R)| \leqslant \left|\xi_{p+q}\left[\bigcup_{k=1}^{r}(D_k\times E_k)\right]\right| = |\xi_{p+q}(R)|.$$

This establishes (I) and finishes the proof for the product $\xi_p^a \times \xi_q^a$. The other products can be treated in exactly the same way, by establishing the inequality (1) for them. ■

The question whether in analogy with (5.25), $\xi_p^a \times \xi_q^a = \xi_{p+q}^a$, has a negative answer, but this is best demonstrated later, cf. theorem 11.13 below.

6. The permutation group and symmetric sets and functions

Our objective in this section is to find out the simplifications that accrue in the preceding theory when the sets involved are symmetric, i.e. belong to $\mathcal{D}_p^{\mathrm{sym}}$, cf. 1.9. This involves an investigation of the action of the appropriate permutation groups on the canonical coefficients $\lambda_\pi^p(D,h)$ and $\gamma_k^p(D,h)$, i.e. an investigation of how $\lambda_\pi^p(D^\phi,h)$, $\gamma_k^p(D^\phi,h)$, and $\lambda_\pi^p(D,h^\psi)$, $\gamma_k^p(D,h^\psi)$ might be related. (Recall that D^ϕ is defined in 1.37.)

It is convenient to first dispose of the much simpler action of the permutation group on the measures ξ_p, ξ_p^a, ξ_p^b. For completeness we also include ℓ_p. We obviously have:

6.1. Triviality. $\forall\phi\in\mathrm{Perm}(p)$, *$\phi$ is ℓ_p and ξ_p measure preserving, i.e.* $\forall\phi\in\mathrm{Perm}(p)$,
(a) $\forall D\in\mathcal{D}_p$, $\ell_p(D^\phi)=\ell_p(D)$ & $\xi_p(D^\phi)=\xi_p(D)$;
(b) $\forall x'\in(\mathcal{L}_2)'$ & $\forall A\in\mathcal{B}_p$, $|x'\circ\xi_p|(A^\phi)=|x'\circ\xi_p|(A)$.

Since the diagonal skeleton I_1^p is symmetric, the corresponding results for ξ_p^a, ξ_p^b easily follow from 6.1, by virtue of the obvious result:

6.2. Triviality. *Let* (i) $\mathcal{X}$ *be a Banach space and* $p\in\mathbb{N}_+$,
(ii) $\rho\in\mathrm{CA}(\mathcal{D}_p,\mathcal{X})$ *be permutation-invariant, i.e.*

$$\forall\phi\in\mathrm{Perm}(p)\ \&\ \forall D\in\mathcal{D}_p,\qquad \rho(D^\phi)=\rho(D),$$

(iii) $\rho_0(\cdot):=\rho(\cdot\cap S)$, *where* $S\in\mathcal{B}_p^{\mathrm{sym}}$.
Then ρ_0 is permutation-invariant.

We thus obtain

$$(6.3)\qquad \begin{cases}\forall\phi\in\mathrm{Perm}(p)\ \&\ \forall D\in\mathcal{D}_p,\quad \xi_p^a(D^\phi)=\xi_p^a(D),\quad \xi_p^b(D^\phi)=\xi_p^b(D),\\ \forall\phi\in\mathrm{Perm}(p),\quad \forall x'\in(\mathcal{L}_2)'\ \&\ \forall A\in\mathcal{B}_p,\\ \qquad |x'\circ\xi_p^a|(A^\phi)=|x'\circ\xi_p^a|(A),\quad |x'\circ\xi_p^b|(A^\phi)=|x'\circ\xi_p^b|(A).\end{cases}$$

We first address the following basic question. Let $p\in\mathbb{N}_+$, $A\subseteq\mathbb{R}^p$, $k\in[0,[p/2]]$, and $\pi\in\Pi_k^p$. Given any $\phi\in\mathrm{Perm}(p)$, is there a $\bar\pi\in\Pi_k^p$ and a $\bar\phi\in\mathrm{Perm}(p-2k)$, such that

$$\forall h\in\mathbb{R}^{p-2k},\qquad (A^\phi)_\pi^p(h)=A_{\bar\pi}^p(h^{\bar\phi})?$$

We proceed to show that the answer is affirmative, when $\bar{\pi}$ and $\bar{\phi}$ are taken as in the following definition:

6.4. *Definition.* Let (i) $p \in \mathbb{N}_+$, $\phi \in \mathrm{Perm}(p)$ and $k \in [0, [p/2]]$,

(ii) $\pi \in \{\{i_1, j_1\}, \dots, \{i_k, j_k\}\} \in \Pi_k^p$, $i_1 < \cdots < i_k$,

$$M'_\pi := [1, p] \backslash M_\pi = \{m_1, \dots, m_{p-2k}\}, \qquad m_1 < \cdots < m_{p-2k},$$

(iii) $\forall \alpha \in [1, k]$, $\bar{\imath}_\alpha := \phi(i_\alpha) \wedge \phi(j_a)$, $\bar{\jmath}_\alpha := \phi(i_\alpha) \vee \phi(j_a)$,

(iv) $\bar{\phi} \in \mathrm{Perm}(p - 2k)$ be such that it yields the increasing rearrangement of $\phi(m_1), \dots, \phi(m_{p-2k})$, i.e. $\bar{\phi}$ be such that

$$\phi(m_{\bar{\phi}(1)}) < \phi(m_{\bar{\phi}(2)}) < \cdots < \phi(m_{\bar{\phi}(p-2k)}).$$

We shall call $\bar{\pi} := \{\{\bar{\imath}_1, \bar{\jmath}_1\}, \dots, \{\bar{\imath}_k, \bar{\jmath}_k\}\} \in \Pi_k^p$, rearranged as per 1.16(b), the *ϕ-distortion of* π, and call $\bar{\phi}$ *the* (ϕ, π)*-permutation of* $[1, p - 2k]$.

Note. For $k = 0$, $\bar{\pi} = \emptyset = \pi$, and $\bar{\phi} = \phi^{-1} \in \mathrm{Perm}(p)$, as the reader can easily check.

Example. Let $p = 9$, $k = 3$, $\phi \in \mathrm{Perm}(9)$ be given by

$$\phi := \begin{pmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9 \\ 9, & 2, & 8, & 1, & 6, & 4, & 3, & 7, & 5 \end{pmatrix}, \qquad \pi := \{\{2, 4\}, \{3, 5\}, \{5, 7\}\} \in \Pi_3^7.$$

Then as the reader can easily check

$$\{\{\bar{\imath}_1, \bar{\jmath}_1\}, \{\bar{\imath}_2, \bar{\jmath}_2\}, \{\bar{\imath}_3, \bar{\jmath}_3\}\} = \{\{1, 2\}, \{4, 8\}, \{3, 6\}\}.$$

Rearranging, we get

$$\bar{\pi} = \{\{1, 2\}, \{3, 6\}, \{4, 8\}\} \in \Pi_3^7.$$

Also since $M'_\pi = \{m_1, m_2, m_3\} = \{1, 8, 9\}$, therefore $\phi(m_1) = 9$, $\phi(m_2) = 7$, $\phi(m_3) = 5$. Hence $\bar{\phi} \in \mathrm{Perm}(3)$, is given by

$$\begin{pmatrix} 1, & 2, & 3 \\ 3, & 2, & 1 \end{pmatrix}.$$

6.5. Basic proposition. *Let $p \in \mathbb{N}_+$ and $k \in [0, [p/2]]$. Then $\forall A \subseteq \mathbb{R}^p$, $\forall \phi \in \mathrm{Perm}(p)$, $\forall \pi \in \Pi_k^p$ and $\forall h \in \mathbb{R}^{p-2k}$,*

$$(A^\phi)_\pi^p(h) = A_{\bar{\pi}}^p(h^{\bar{\phi}}),$$

where $\bar{\pi}$ is the ϕ-distortion of π and $\bar{\phi}$ is the (ϕ, π)-permutation of $[1, p - 2k]$.

Proof. Case 1. Let $k \geqslant 1$. Let $\pi \in \Pi_k^p$ and M'_π be given by 6.4(ii), $A \subseteq \mathbb{R}^p$ and $\phi \in \mathrm{Perm}(p)$.

Given $\tau \in \mathbb{R}^k$ and $h \in \mathbb{R}^{p-2k}$, define $t \in \mathbb{R}^p$ by

$$(1) \qquad t_{i_\alpha} = \tau_i = t_{j_\alpha} \quad \& \quad \forall t_{m_\beta} = h^\beta, \qquad \forall \alpha \in [1, k] \quad \& \quad \forall \beta \in [1, p - 2k].$$

Then by definitions 4.10 and (4.8),

$$(2) \ \tau \in (A^\phi)_\pi^p(h) \quad \Longleftrightarrow \quad t \in A^\phi \quad \Longleftrightarrow \quad s := t^{\phi^{-1}} = (t_{\phi^{-1}(1)}, \dots, t_{\phi^{-1}(p)}) \in A.$$

From (1) we see that

$$(3) \qquad \forall \alpha \in [1, k], \quad s_{\phi(i_\alpha)} = t_{\phi^{-1}\{\phi(i_\alpha)\}} = t_{i_\alpha} = \tau_\alpha = t_{j_\alpha} = t_{\phi^{-1}\{\phi(j_\alpha)\}} = s_{\phi(j_\alpha)}.$$

But for the cells $\{\bar{\imath}_\alpha, \bar{\jmath}_\alpha\}$ of $\bar{\pi}$, the ϕ-distortion of π, we know that $\bar{\imath}_\alpha$, $\bar{\jmath}_\alpha$, are the smaller and greater of $\phi(i_\alpha)$, $\phi(j_\alpha)$. Hence from (3) we infer that

$$(4) \qquad \forall \alpha \in [1,k], \qquad s_{\bar{\imath}_\alpha} = \tau_\alpha = s_{\bar{\jmath}_\alpha}.$$

Next, since each $\bar{\imath}_\alpha$, $\bar{\jmath}_\alpha$ is one of $\phi(i_\sigma)$, $\phi(j_\alpha)$ therefore

$$M_{\bar{\pi}} = \{\phi(i_\alpha), \phi(j_\alpha), \ldots, \phi(i_k), \phi(j_k)\}.$$

Hence

$$M'_{\bar{\pi}} := [1,p]\backslash M_{\bar{\pi}} = \{\phi(m_1), \ldots, \phi(m_{p-2k})\}$$
$$= \{\bar{m}_1, \bar{m}_2, \ldots, \bar{m}_{p-2k}\} \quad \text{where} \quad \bar{m}_1 < \cdots < \bar{m}_{p-2k},$$

say. Then for the distortion $\bar{\phi}$ we know that for $\beta \in [1, p-2k]$, $\bar{m}_\beta = \phi(m_{\bar{\phi}(\beta)})$. Hence $m_{\bar{\phi}(\beta)} = \phi^{-1}(\bar{m}_\beta)$, and therefore

$$(5) \qquad \forall \beta \in [1, p-2k], \qquad s_{\bar{m}_\beta} = t_{\phi^{-1}(\bar{m}_\beta)} = t_{m_{\bar{\phi}(\beta)}} = h^{\bar{\phi}(\beta)}, \quad \text{by (1)}.$$

We see from (4), (5) and (4.8) that

$$\tau \in A^p_{\bar{\pi}}(h^{\bar{\phi}}) \iff s \in A \iff \tau \in (A^\phi)^p_\pi(h), \quad \text{by (2)}.$$

The two sets are thus equal.

Case 2. Let $k = 0$. Then since $\bar{\pi} = \emptyset = \pi$ and $\bar{\phi} = \phi^{-1}$, the equality easily follows from the expression for $(A^p_\emptyset)(h)$ given in the note to 4.6. ■

We next address the following question. Let $p \in \mathbb{N}_+$, $k \in [1, [p/2]]$, and $\pi \in \Pi^p_k$. Given any $\psi \in \text{Perm}(p-2k)$, is there a $\phi \in \text{Perm}(p)$ such that $\forall D \in \mathcal{D}_p$ and $\forall h \in \mathbb{R}^{p-2k}$,

$$\lambda^p_\pi(D, h^\psi) = \lambda^p_\pi(D^\phi, h)?$$

The affirmative answer (cf. 6.9) depends on the following definition and lemma:

6.6. *Definition.* (π-extension from Perm$(p-2k)$ to Perm(p)) Let
(i) $p \in \mathbb{N}_+$, $k \in [1, [p/2]]$, $M \subseteq [1,p]$, $\#M = 2k$, and $\psi \in \text{Perm}(p-2k)$,
(ii) $M' := [1,p]\backslash M = \{m_1, \ldots, m_{p-2k}\} \subseteq [1,p]$, $m_1 < \cdots < m_{p-2k}$.
Then (*a*) the *M-extension* $\bar{\psi}_M$ (briefly $\bar{\psi}$) in Perm(p) of ψ is given by

$$\forall i \in M, \qquad \bar{\psi}(i) := i \quad \& \quad \forall \alpha \in [1, p-2k], \qquad \phi(m_\alpha) := m_{\psi^{-1}(\alpha)};$$

(*b*) $\forall \pi \in \Pi^p_k$, the *π-extension* $\bar{\psi}_\pi$ of ψ is by definition $\bar{\psi}_{M_\pi}$.

Example. Let $p = 12$, $k = 4$, $M = \{1, 2, 4, 5, 8, 9, 10, 11\} \subseteq [1, 12]$, $\psi \in \text{Perm}(4)$ be given by

$$\psi(1) = 2, \qquad \psi(2) = 4, \qquad \psi(3) = 3, \qquad \psi(4) = 1.$$

Then the M-extension $\bar{\psi} \in \text{Perm}(12)$ of ψ is given on M, by

$$\bar{\psi}(i) = i, \qquad \forall i \in M.$$

As for $\bar{\psi}$ on $M' = \{3, 6, 7, 12\} = \{m_1, m_2, m_3, m_4\}$ say, we have

$$\forall \alpha \in [1, 4], \qquad \bar{\psi}(m_\alpha) = m_{\psi^{-1}(\alpha)}.$$

Since $\psi^{-1}(1) = 4$, $\psi^{-1}(2) = 1$, $\psi^{-1}(3) = 3$, $\psi^{-1}(4) = 2$, we see that

$$\bar{\psi}(3) = \bar{\psi}(m_1) := m_{\psi^{-1}(1)} = m_4 = 12.$$

Similarly, $\bar\psi(6) = 3$, $\bar\psi(7) = 7$, $\bar\psi(12) = 6$. Thus $\bar\psi$ leaves each point of M fixed, but on M', $\bar\psi$ carries 3, 6, 7, 12 to 12, 3, 7, 6, respectively.

Since ψ^{-1} has the same parity as ψ, and the identity permutation I on M has even parity, it follows easily that $\forall M \subseteq [1, p]$ & $\# M = 2k$ & $\forall \psi \in \mathrm{Perm}(p - 2k)$,

(6.7) ψ and $\bar\psi_M$ have the same parity.

6.8. Lemma. *Let* (i) $p \in \mathbb{N}_+$, $k \in [1, [p/2]]$, (ii) $\pi \in \Pi_k^p$, (iii) $\psi \in \mathrm{Perm}(p - 2k)$ *and* (iv) ϕ *be the* π*-extension of* ψ. *Then*

$$\forall A \subseteq \mathbb{R}^p \quad \& \quad \forall h \in \mathbb{R}^{p-2k}, \qquad (A^\phi)^p_\pi(h) = A^p_\pi(h^\psi).$$

Proof. Let $A \subseteq \mathbb{R}^p$ and $h \in \mathbb{R}^{p-2k}$. For ϕ as in (iv), let $\bar\pi$ be the ϕ-distortion of π and $\bar\phi \in \mathrm{Perm}(p - 2k)$ be the (ϕ, π)-permutation of $[1, p - 2k]$, cf. 6.4. Grant momentarily that

(I) $$\bar\pi = \pi \quad \& \quad \bar\phi = \psi.$$

Then by proposition 6.5 and (I),

$$(A^\phi)^p_\pi(h) = A^p_{\bar\pi}(h^{\bar\phi}) = A^p_\pi(h^\psi),$$

as desired. Hence it only remains to show (I).

Proof of (I). Let $\pi = \{\{i_1, j_1\}, \ldots, \{i_k, j_k\}\}$. Since by (iv) and 6.6, $\phi := \bar\psi_{M_{\bar\pi}} = I$ on M_π, therefore $\forall \alpha \in [1, k]$,

$$\bar\imath_\alpha = \phi(i_\alpha) \wedge \phi(\bar\jmath_\alpha) = i_\alpha \quad \& \quad \bar\jmath_\alpha = \phi(i_\alpha) \vee \phi(\bar\jmath_\alpha) = j_\alpha,$$

i.e. $\{\bar\imath_\alpha, \bar\jmath_\alpha\} = \{i_\alpha, j_\alpha\}$. Thus $\bar\pi = \pi$.

Next let $M'_\pi = \{m_1, \ldots, m_{p-2k}\}$, $m_1 < \cdots < m_{p-2k}$. Then by 6.6(a), $\forall \beta \in [1, p - 2k]$,

$$\phi\{m_{\psi(\beta)}\} = m_{\psi^{-1}[\psi(\beta)]} = m_\beta.$$

The chain $m_1 < \cdots < m_{p-2k}$ thus entails:

$$\phi\{m_{\psi(1)}\} < \cdots < \phi\{m_{\psi(p-2k)}\}.$$

This shows, cf. 6.4(iv), that $\bar\phi = \psi$. Thus (I). ■

Using definitions 4.10, 4.13, and lemma 6.8, we easily get the answer to the question raised at the outset:

6.9. Proposition. *Let* p, k, ψ, π *be as in 6.8, and* $\bar\psi_\pi$ *be the* π*-extension of* ψ. *Then* $\forall D \in \mathcal{D}_p$ & $\forall h \in \mathbb{R}^{p-2k}$,

(a) $\lambda^p_\pi(D, h^\psi) = \lambda^p_\pi(D^{\bar\psi_\pi}, h)$;

(b) $$\gamma^p_k(D, h^\psi) := \sum_{\pi \in \Pi^p_k} \lambda^p_\pi(D, h^\psi) = \sum_{\pi \in \Pi^p_k} \lambda^p_\pi(D^{\bar\psi_\pi}, h).$$

For symmetric D, we immediately get from 6.9(a) and 6.9(b) the following corollary:

6.10. Corollary. *Let* $p \in \mathbb{N}_+$, $k \in [1, [p/2]]$, *and* $\pi \in \Pi^p_k$. *Then*

(a) $\forall D \in \mathcal{D}^{\mathrm{sym}}_p$, $\lambda^p_\pi(D, \cdot)$ *is symmetric on* $\mathbb{R}^{p-2k}$;

(b) $\forall D \in \mathcal{D}^{\mathrm{sym}}_p$, $\gamma^p_k(D, \cdot)$ *is symmetric on* $\mathbb{R}^{p-2k}$.

The most convenient among the partitions in Π_k^p, is the one for which $M_\pi = [1, 2k]$, and the cells occur in natural order:

(6.11) $$\forall k \in \mathbb{N}_+, \qquad \pi_k := \{\{1,2\},\{3,4\},\ldots,\{2k-1,2k\}\}.$$

We shall refer to π_k as the *k-standard partition.*

A permutation $\phi \in \operatorname{Perm}(p)$ that is especially useful is the one for which the ϕ-distortion of $\pi \in \Pi_k^p$ is the k-standard partition π_k, and which also preserves the ordering on the set M'_π. This is defined as follows:

6.12. *Definition.* Let (i) $p \in \mathbb{N}_+$ and $k \in [1, [p/2]]$,
(ii) $\pi = \{\{i_1, j_1\}, \ldots, \{i_k, j_k\}\} \in \Pi_k^p$, $i_1 < \cdots < i_p$.
Then ϕ_π on $[1, p]$ is defined by

$$\forall \alpha \in [1,k], \qquad \phi_\pi(i_\alpha) = 2\alpha - 1, \qquad \phi_\pi(j_\alpha) = 2\alpha$$

$$\&\ \phi_\pi \text{ is increasing on } M'_\pi \text{ onto } [2k+1, p].$$

Note. $\phi_\pi({}^*\pi) = {}^*\pi_k$, $\phi_\pi(\pi^*) = \pi_k^*$ & ϕ_π is increasing on ${}^*\pi$.

It is necessary to record the deformations, cf. 6.4, caused by the permutation ϕ_π just defined; we leave its easy proof to the reader:

6.13. Lemma. *Let* (i) $p \in \mathbb{N}_+$, $k \in [0, [p/2]]$ *and* $\pi \in \Pi_k^p$.
(ii) $\bar\pi$ *be the* ϕ_π*-deformation of* π *and* $\bar\phi$ *be the* (ϕ, π) *permutation of* $[1, p-2k]$. *Then* (*a*) $\bar\pi = \pi_k$ & $\forall \beta \in [1, p-2k]$, $\bar\phi(\beta) = \beta$;
(*b*) $I(\pi, p)^{\phi_\pi^{-1}} = I(\pi_k, p)$.

This lemma allows us to conclude immediately from the basic proposition 6.5:

6.14. Corollary. *Let* $p \in \mathbb{N}_+$, $k \in [0, [p/2]]$, $\pi \in \Pi_k^p$, *and* ϕ_π *be as in 6.12. Then*

$$\forall A \subseteq \mathbb{R}^p \quad \& \quad \forall h \in \mathbb{R}^{p-2k}, \qquad (A^{\phi_\pi})_\pi^p(h) = A_{\pi_k}^p(h).$$

Combining 6.14, 6.9(*a*) and definitions 4.10, 4.13, we get the following useful analogue of proposition 6.9 for the k-standard partition:

6.15. Proposition. *Let* (i) $p \in \mathbb{N}_+$, $k \in [1, [p/2]]$, *and* $\pi \in \Pi_k^p$, (ii) ϕ_π *be as in 6.12,* (iii) $\psi \in \operatorname{Perm}(p-2k)$ & $\bar\psi := \bar\psi_{\pi_k}$ *be the* π_k*-extension of* ψ, *cf. definition 6.6*(*b*). *Then* $\forall D \in \mathcal{D}_p$ *and* $\forall h \in \mathbb{R}^{p-2k}$,

$$\lambda_\pi^p(D, h) = \lambda_{\pi_k}^p(D^{\phi_\pi^{-1}}, h) \quad \& \quad \lambda_\pi^p(D, h^\psi) = \lambda_{\pi_k}^p(D^{\bar\psi\phi_\pi^{-1}}, h).$$

The corresponding results for the coefficients γ_k^p are, cf. 6.9(*b*):

(6.16) $$\left\{\begin{aligned} & p \in \mathbb{N}_+, \quad \forall k \in [1,[p/2]], \quad \forall D \in \mathcal{D}_p, \quad \forall h \in \mathbb{R}^{p-2k}, \\ & \forall \psi \in \operatorname{Perm}(p-2k)\ \&\ \forall \bar\psi := \bar\psi_{[1,2k]}, \\ & \qquad \gamma_k^p(D,h) := \sum_{\pi \in \Pi_k^p} \lambda_{\pi_k}^p(D^{\phi_\pi^{-1}}, h), \\ & \qquad \gamma_k^p(D,h^\psi) := \sum_{\pi \in \Pi_k^p} \lambda_{\pi_k}^p(D^{\bar\psi\phi_\pi^{-1}}, h). \end{aligned}\right.$$

For symmetric D, once again 6.15 and (6.16), together with 6.10, readily yield the result that $\lambda^p_\pi(D, h^\psi)$ is the same for all π and for all ψ, and that $\gamma^p_k(D, h^\psi)$ is a constant multiple of $\lambda^p_{\pi_k}(D, h)$:

6.17. Corollary. *Let $p \in \mathbb{N}_+$, $k \in [1, [p/2]]$ & $\pi \in \Pi^p_k$. Then $\forall D \in \mathcal{D}^{\mathrm{sym}}_p$ & $\forall h \in \mathbb{R}^{p-2k}$, and $\forall \psi \in \mathrm{Perm}(p-2k)$,*

$$\lambda^p_\pi(D, h^\psi) = \lambda^p_{\pi_k}(D, h), \qquad \gamma^p_k(D, h^\psi) = \binom{p}{2k}\alpha_{2k}\lambda^p_{\pi_k}(D, h).$$

The second equality in 6.17 brings about some simplification in the formulae for the covariance kernels $\Gamma^{pq}_k(D, E)$, cf. 5.1, when D or E or both are symmetric, and this in turn simplifies the formula for the covariance $(\xi_p(D), \xi_q(D))$. We have:

6.18. Corollary. *Let* (i) *$p, q \in \mathbb{N}_+$, be such that $p + q = 2r$ is even and $q \leqslant p$,* (ii) *$k \in [0, [q/2]]$. Then*

(*a*) $\forall D \in \mathcal{D}_p$ & $\forall E \in \mathcal{D}^{\mathrm{sym}}_q$,

$$\Gamma^{pq}_k(D, E) = (q-2k)!\binom{q}{2k}\alpha_{2k}\int_{\mathbb{R}^{q-2k}}\gamma^p_{\frac{1}{2}(p-q)+k}(D, h)\gamma^p_{\pi_k}(E, h)\ell_{q-2k}\,(\mathrm{d}h);$$

(*b*) $\forall D \in \mathcal{D}^{\mathrm{sym}}_p$ & $\forall E \in \mathcal{D}_q$,

$$\Gamma^{pq}_k(D, E) = \sum_{\phi \in \mathrm{Perm}(q-2k)}\binom{p}{q-2k}\alpha_{p-q+2k} \times \int_{\mathbb{R}^{q-2k}}\lambda^p_{\pi_{\frac{1}{2}(p-q)+k}}(D, h)\gamma^q_k(E, h^\phi)\ell_{q-2k}\,(\mathrm{d}h);$$

(*c*) $\forall D \in \mathcal{D}^{\mathrm{sym}}_p$ & $\forall E \in \mathcal{D}^{\mathrm{sym}}_q$,

$$\Gamma^{pq}_k(D, E) = (q-2k)!\binom{p}{q-2k}\binom{q}{2k}\alpha_{p-q+2k}\alpha_{2k} \times \int_{\mathbb{R}^{p-2k}}\lambda^p_{\pi_{\frac{1}{2}(p-q)+k}}(D, h)\gamma^q_{\pi_k}(E, h^\phi)\ell_{q-2k}\,(\mathrm{d}h).$$

Proof. Let $D, E \in \mathcal{D}_p$. Then by 5.1

$$\Gamma^{pq}_k(D, E) := \sum_{\phi \in \mathrm{Perm}(q-2k)}\int_{\mathbb{R}^{q-2k}}\gamma^p_{\frac{1}{2}(p-q)+k}(D, h)\gamma^p_k(E, h^\phi)\ell_{q-2k}\,(\mathrm{d}h).$$

The results follow when one or both factors in the integrand are simplified in the light of the second equality in corollary 6.17. ∎

Another important topic is that of the action on $L_2(\mathbb{R}^p)$ of the symmetrization operator $f \mapsto \tilde{f}$, cf. definition 1.39(*b*) and (1.46). The following useful result is a simple consequence of the permutation invariance of the measure ℓ_p, cf. (6.1):

6.19. Lemma. (Symmetrization on $L_2(\mathbb{R}^p)$) $\forall f, g \in L_2(\mathbb{R}^p)$,

$$(\tilde{f}, g) = (f, \tilde{g}) = (\tilde{f}, \tilde{g}).$$

Thus on letting $\forall f \in L_2(\mathbb{R}^p)$, $S(f) := \tilde{f}$, we see that S is an orthogonal projection on $L_2(\mathbb{R}^p)$ onto $L^{\mathrm{sym}}_2(\mathbb{R}^p)$, the null space of which includes all antisymmetric functions on $\mathbb{R}^p$.

It follows from 6.19 that $|\tilde{f}|_{2,\ell_p} \leqslant |f|_{2,\ell_p}$, for all $f \in L_2(\mathbb{R}^p)$. Since for measurable $f \notin L_2(\mathbb{R}^p)$, $|f|_{2,\ell_p} = \infty$, we see at once that

$$(6.20) \qquad \forall f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1), \qquad |\tilde{f}|_{2,\ell_p} \leqslant |f|_{2,\ell_p} \leqslant \infty.$$

There is, however, a more delicate reverse inequality to (6.20) that rests on the ℓ_p-negligibility of the diagonal skeleton I_1^p, cf. (4.14). This requires first of all the following simple results on the half-spaces S_ϕ^p, $\phi \in \text{Perm}(p)$ into which $\mathbb{R}^p \backslash I_1^p$ is dissected, cf. 4.2(b), the easy proof of which we omit:

$$(6.21) \qquad \begin{cases} \forall p \in \mathbb{N}_+ \ \& \ \forall \phi, \psi, \psi' \in \text{Perm}(p), \\ \psi^{-1}(S_\phi^p) = (S_\phi^p)^{\psi^{-1}} = S_{\psi\phi}^p \quad \& \quad (\chi_{S_\phi^p})^\psi = \chi_{S_{\psi\phi}^p}, \\ \qquad\qquad\qquad \text{cf. definitions 1.37 and 1.42;} \\ \psi \neq \psi' \Longrightarrow (\chi_{S_\phi^p})^\psi \cdot (\chi_{S_\phi^p})^{\psi'} = 0 \quad \text{on} \quad \mathbb{R}^p. \end{cases}$$

The cherished reverse inequality links the L_2-norm of f with the L_2-norms of the symmetrizations of the restrictions of f:

6.22. Lemma. (Symmetrization inequality) *Let* (i) $p \in \mathbb{N}_+$, (ii) $f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$. *Then*

$$(1/p!)|f|_{2,\ell_p} \leqslant \sup_{A \in \mathcal{B}_p} |(f\chi_A)\tilde{}\,|_{2,\ell_p} \in [0, \infty].$$

Proof. Let S_ϕ^p, briefly S_ϕ, for $\phi \in \text{Perm}(p)$, be the diagonal half-spaces of $\mathbb{R}^p$, cf. 4.2(b). Grant momentarily that

$$(\text{I}) \qquad \forall \phi \in \text{Perm}(p), \qquad \frac{1}{p!}\int_{S_\phi} |f(t)|^2 \ell_p\,(\text{d}t) = |(f\chi_{S_\phi})\tilde{}\,|_{2,\ell_p}^2 \in [0, \infty].$$

Then since by (4.14), $I_1^p \in \mathcal{N}_{\ell_p}$ and, cf. 4.2(b), $\mathbb{R}^p \backslash I_1^p = \bigcup_{\phi \in \text{Perm}(p)} S_\phi$ & S_ϕ are $\|$,

$$\begin{aligned} |f|_{2,\ell_p}^2 &= \int_{\mathbb{R}^p \backslash I_1^p} |f(t)|^2 \ell_p\,(\text{d}t) = \sum_{\phi \in \text{Perm}(p)} \int_{S_\phi} |f(t)|^2 \ell_p\,(\text{d}t) \\ &= p! \sum_{\phi \in \text{Perm}(p)} |(f\chi_{S_\phi})\tilde{}\,|_{2,\ell_p}^2, \qquad \text{by (I)} \\ &\leqslant p! \sum_{\phi \in \text{Perm}(p)} \sup_{A \in \mathcal{B}_p} |(f\chi_A)\tilde{}\,|_{2,\ell_p}^2 = (p!)^2 \sup_{A \in \mathcal{B}_p} |(f\chi_A)\tilde{}\,|_{2,\ell_p}^2. \end{aligned}$$

Square-rooting and dividing by $p!$, we get the desired result. Here it only remains to prove (I).

Proof of (I). Since neither f nor χ_{S_ϕ} are in $L_2(\mathbb{R}^p)$, we are barred from using the nice inner product notation. Instead we have to proceed by noting that for $t \in \mathbb{R}^p$,

$$\begin{aligned} p!(f\chi_{S_\phi})\tilde{}\,(t) &:= \sum_{\psi \in \text{Perm}(p)} (f\chi_{S_\phi})(t^\psi) \\ &= \sum_{\psi \in \text{Perm}(p)} f(t^\psi)\chi_{S_\phi}(t^\psi). \end{aligned}$$

Let $\{\psi_1, \ldots, \psi_{p!}\}$ be a denumeration of Perm(p). Then

$$\begin{aligned}(p!)^2[(f\chi_{S_\phi})\tilde{}\,(t)]^2 &= \left[\sum_{i=1}^{p!} f(t^{\psi_i})\chi_{S_\phi}(t^{\psi_i})\right]^2 \\ &= \sum_{i=1}^{p!} f(t^{\psi_i})^2\chi_{S_\phi}(t^{\psi_i})^2 + \sum_{i\neq j} f(t^{\psi_i})\chi_{S_\phi}(t^{\psi_i})f(t^{\psi_j})\chi_{S_\phi}(t^{\psi_j}) \\ &= \sum_{i=1}^{p!}\{f(t^{\psi_i})\chi_{S_\phi}(t^{\psi_i})\}^2 + 0, \\ &\qquad\qquad \text{since } \chi_{S_\phi}(t^{\psi_i})\chi_{S_\phi}(t^{\psi_j}) = 0, \text{ by (6.21).}\end{aligned}$$

Hence integrating over $\mathbb{R}^p$,

$$\begin{aligned}(p!)^2|(f\chi_{S_\phi})\tilde{}\,|^2_{2,\ell_p} &:= \sum_{i=1}^{p!}\int_{\mathbb{R}^p}\{(f\cdot\chi_{S_\phi})(t^{\psi_i})\}^2\ell_p\,(\mathrm{d}t) \\ &= \sum_{i=1}^{p!}\int_{\mathbb{R}^p}\{(f\cdot\chi_{S_\phi})(t)\}^2\ell_p\,(\mathrm{d}t), \qquad \text{since } \ell_p \text{ is } \psi_i \text{ invariant,} \\ &= \sum_{i=1}^{p!}|f\chi_{S_\phi}|^2_{2,\ell_p} = p!\,|f\chi_{S_\phi}|^2_{2,\ell_p}.\end{aligned}$$

Dividing by $(p!)^2$, we get (I). ■

7. The Lebesgue negligibility of the diagonal skeletons and of the canonical coefficients

To advance further in the study of the measure ξ_p we have to study all the diagonal skeletons I^p_k, for $k \in [1,[p/2]]$, cf. (4.5). Even though $I^p_k \notin \mathcal{D}_p$, we have $\forall D \in \mathcal{D}_p$, $D \cap I^p_k \in \mathcal{D}_p$, and the coefficients $\gamma^p_k(D\cap I^p_j,\cdot)$ are well-defined functions on $\mathbb{R}^{p-2k}$. Our objective in this section is to show that

$$(*) \qquad \forall \text{ even } p \in \mathbb{N}_+ \quad \& \quad \forall k \in [0, p/2-1], \qquad \operatorname{supp}\gamma^p_k(D\cap I^p_{p/2},\cdot) \in \mathcal{N}_{\ell_{p-2k}},$$

i.e. that all except the last of the coefficients $\gamma^p_k(D\cap I^p_{p/2},\cdot)$ vanish a.e. (ℓ_{p-2k}) on $\mathbb{R}^{p-2k}$, cf. (7.6). The last (i.e. $k = p/2$) coefficient is always $\mathbb{E}_{\mathbb{P}}\{\xi_p(D)\}$, i.e. is constant-valued, cf. 5.9. The consequences of the result $(*)$ are crucial to the remaining sections.

We begin by recording some obvious properties of the skeletons defined in (4.5):

7.1. Triviality. *Let* $p \in \mathbb{N}_+$ *&* $\forall k \in [0,[p/2]]$. *Then*

(*a*) *for* $0 \leqslant k \leqslant \bar{k} \leqslant [p/2]$, $I^p_{[p/2]} \subseteq I^p_{\bar{k}} \subseteq I^p_k \subseteq I^p_1 \subseteq I^p_0 = \mathbb{R}^p$;

(*b*)

$$I^p_{[p/2]} = \begin{cases} \displaystyle\bigcup_{\pi\in\Pi_{[1,p]}} I(\pi,p), & \textit{for even } p; \\ \displaystyle\bigcup_{i=1}^{p}\bigcup_{\pi\in\Pi_{[1,p]\setminus[i]}} I(\pi,p), & \textit{for odd } p; \end{cases}$$

(*c*) I_k^p *is symmetric;*
(*d*) $\forall k \in [1, [p/2]],\ \ I_k^p \in \mathcal{N}_{\ell_p}$;
(*e*) $\forall p \in \mathbb{N}_+,\ \ I_k^p \times \mathbb{R}^q \subseteq I_k^{p+q}$.

The proof, which is quite routine, is omitted. More difficult and important is the next lemma, to state which it is convenient to have a notation for the affine displacements of the coordinate hyperplanes of $\mathbb{R}^p$, stemming from a vector h in $\mathbb{R}^{p-2k}$:

7.2. *Notation.* Let $p \in \mathbb{N}_+$ & $k \in [1, [p/2]]$. Then $\forall h \in \mathbb{R}^{p-2k}$, $\forall \alpha \in [1, k]$ & $\forall \beta \in [1, p-2k]$,

$$[h]^{p,k}_{\alpha,\beta} := \{\tau : \tau = (\tau^1, \dots, \tau^k) \in \mathbb{R}^k \ \&\ \tau^\alpha = h^\beta\} \subseteq \mathbb{R}^k,$$

i.e. $[h]^{p,k}_{\alpha,\beta}$ is the affine hyperplane of $\mathbb{R}^k$, parallel to the coordinate hyperplane $\{\tau : \tau \in \mathbb{R}^k \ \&\ \tau^\alpha = 0\}$, obtained by translating the latter along its normal a distance h_β. Obviously $\dim[h]^{p,k}_{\alpha,\beta} = k-1$. For $k = 0$, we define $[h]^{p,0}_{\alpha,\beta} := \emptyset$.

For each α there will be $p - 2k$ such parallel hyperplanes. The total number as α ranges over $[1, k]$ is $k(p-2k)$.[8]

Example. For $p = 7$ and $k = 3$, we have $h \in \mathbb{R}^1$, and so $k(p-2k) = 3$. It is easily checked that the three affine hyperplanes $[h]^{7,3}_{\alpha,1}$ of $\mathbb{R}^3$ for $\alpha = 1, 2, 3$, are the planes $\{x = h\}$, $\{y = h\}$, $\{z = h\}$ in $\mathbb{R}^3$.

7.3. Main lemma. *Let* (i) $p \in \mathbb{N}_+$ & $k \in [0, [p/2]]$,
(ii) $\pi \in \Pi_k^p$ & $\bar\Delta := \{\bar\imath, \bar\jmath\} \subseteq [1, p]$ & $\bar\Delta \notin \pi$.
Then (*a*) $\forall h \in \mathbb{R}_*^{p-2k} := \mathbb{R}^{p-2k} \backslash I_1^{p-2k}$,

$$(I^p_{\bar\Delta})^p_\pi \subseteq I_1^k \cup \bigcup_{\alpha=1}^{k} \bigcup_{\beta=1}^{p-2k} [h]^p_{\alpha,\beta} \quad \in \quad \mathcal{N}_{\ell_k};$$

(*b*) $\forall h \in \mathbb{R}_*^{p-2k}$ & $\forall D \in \mathcal{D}_p$, $\lambda^p_\pi(D \cup I^p_{\bar\Delta}, h) = 0$.

Proof. (*a*) Let first $k \in [1, [p/2]]$,

(1) $$\pi = \{\{i_1, j_1\}, \dots, \{i_k, j_k\}\} \in \Pi_k^p \quad \& \quad h = (h^1, \dots, h^{p-2k}) \in \mathbb{R}_*^{p-2k}.$$

We first show the inclusion in the result, to wit

(I) $$A := (I^p_{\bar\Delta})^p_\pi(h) := \wp_{\pi^*}\{(I^p_{\bar\Delta}) \cap I^p_\pi(h)\} \subseteq I_1^k \cup \bigcup_{\alpha=1}^{k} \bigcup_{\beta=1}^{p-2k} [h]^p_{\alpha,\beta}.$$

Proof of (I). Let $\tau = (\tau^1, \dots, \tau^k) \in A$. Then

(2) $$\tau = \wp_{\pi^*}(t) \quad \text{where} \quad t \in I^p_{\bar\Delta} \cap I^p_\pi(h).$$

Since $t \in I^p_\pi(h)$, therefore, cf. (ii),

(3) $$\forall \alpha \in [1, k], \qquad \tau^\alpha = t_{i_\alpha} = t_{j_\alpha} \quad \& \quad \forall \beta \in [1, p-2k], \qquad t_{m_\beta} = h^\beta.$$

But since by (2), $t \in I^p_{\bar\Delta}$, therefore, cf. (ii),

(4) $$t_{\bar\imath} = t_{\bar\jmath}.$$

[8] For p even and $k = p/2$, this number is zero, and the symbol $[h]^{p,k}_{\alpha,\beta}$ is meaningless.

Now let

(5) $$M'_\pi = [1,p]\backslash M_\pi := \{m_{\beta_1}, \dots, m_{\beta_{p-2k}}\}, \qquad m_{\beta_1} < \cdots < m_{\beta_{p-2k}}.$$

Then we have four possible cases depending on whether both or none or just one of $\bar{\imath}$, $\bar{\jmath}$ are in M'_π.

Case 1. Let $\bar{\imath}$ & $\bar{\jmath} \in M'_\pi$. Then by (5), $\exists \beta_1, \beta_2 \in [1, p-2k]$ such that $\bar{\imath} = m_{\beta_1}$ & $\bar{\jmath} = m_{\beta_2}$. Hence by (3) and (4),

$$h^{\beta_1} = t_{m_{\beta_1}} = t_{\bar{\imath}} = t_{\bar{\jmath}} = t_{m_{\beta_2}} = h^{\beta_2},$$

and so $h = (h^1, \dots, h^{\beta_1}, \dots, h^{\beta_2}, \dots, h^{p-2k}) \in I_1^{p-2k}$. But by (1), $h \notin I_1^{p-2k}$. Hence $\bar{\Delta} \cap I_\pi^p(h) = \emptyset$ and so $A = \emptyset$.

Case 2. Let $\bar{\imath}$ & $\bar{\jmath} \in M_\pi$. Then $\exists \alpha_1, \alpha_2 \in [1, k]$ such that $\bar{\imath} \in \{i_{\alpha_1}, j_{\alpha_1}\}$ & $\bar{\jmath} \in \{i_{\alpha_2}, j_{\alpha_2}\}$. Therefore by (3) and (4),

$$\tau^{\alpha_1} = t_{i_{\alpha_1}} = t_{j_{\alpha_1}} = t_{\bar{\imath}} = t_{\bar{\jmath}} = t_{i_{\alpha_2}} = t_{j_{\alpha_2}} = \tau^{\alpha_2}.$$

Here $\alpha_1 \neq \alpha_2$, for were $\alpha_1 = \alpha_2$, then $\bar{\Delta} = \{\bar{\imath}, \bar{\jmath}\} = \{i_{\alpha_1}, j_{\alpha_1}\} \in \pi$, in contradiction to (ii). Thus $\tau = (\tau^1, \dots, \tau^{\alpha_1}, \dots, \tau^{\alpha_2}, \dots, \tau^k) \in I_1^k$.

Case 3. Let $\bar{\imath} \in M_\pi$ & $\bar{\jmath} \in M'_\pi$. Then again $\exists \alpha \in [1, k]$ such that $\bar{\imath} \in \{i_\alpha, j_\alpha\}$ and $\exists \beta \in [1, p-2k]$ such that $\bar{\jmath} = m_\beta$. If $\bar{\imath} = i_\alpha$, then by (3), (4) and (5),

$$\tau^\alpha = t_{i_\alpha} = t_{\bar{\imath}} = t_{\bar{\jmath}} = t_{m_\beta} = h^\beta.$$

If $\bar{\imath} = j_\alpha$, then again by (3), (4) and (5),

$$\tau^\alpha = t_{j_\alpha} = t_{\bar{\imath}} = t_{\bar{\jmath}} = t_{m_\beta} = h^\beta.$$

Thus $\tau = (\tau^1, \dots, \tau^\alpha, \dots, \tau^k)$ and $\tau^\alpha = h^\beta$; i.e. $\tau \in [h]_{\alpha,\beta}^{p,k}$.

Case 4. Let $\bar{\imath} \in M'_\pi$ & $\bar{\jmath} \in M_\pi$. Then again $\exists \beta \in [1, p-2k]$ such that $\bar{\imath} = m_\beta \in M'_\pi$ and $\exists \alpha \in [1, k]$ such that $\bar{\jmath} \in \{i_\alpha, j_\alpha\}$. Hence, by the arguments used in Case 3,

$$\tau^\alpha = t_{i_\alpha} = t_{j_\alpha} = t_{\bar{\imath}} = t_{\bar{\jmath}} = t_{m_\beta} = h^\beta.$$

Thus $\tau = (\tau^1 \cdots \tau^\alpha \cdots \tau^k) \in [h]_{\alpha,\beta}^{p,k}$.

Combining the four cases we see that

$$\tau \in I_1^k \cup \bigcup_{\alpha=1}^{k} \bigcup_{\beta=1}^{p-2k} [h]_{\alpha,\beta}^{p,k}.$$

As this holds $\forall \tau \in A$, we have (I).

Since by 7.1(d), $I_1^k \in \mathcal{N}_{\ell_k}$, and each $[h]_{\alpha,\beta}^{p,k}$, being of dimension $k-1$, is also ℓ_k-negligible, hence so is the union on the RHS of (I). This establishes the result (a) for $k \in [1, p]$.

Finally we consider the case $p \in \mathbb{N}_+$ and $k = 0$. Then for $\pi \in \Pi_0^p$, $\pi = \emptyset$. Consequently, by the note to 4.10,

$$\forall h \in \mathbb{R}^p, \qquad (I_{\bar{\Delta}}^p)_\emptyset^p(h) = \begin{cases} \{0\}, & h \in I_{\bar{\Delta}}^p; \\ \emptyset, & h \in \mathbb{R}^p \backslash I_{\bar{\Delta}}^p. \end{cases}$$

But $I_{\bar{\Delta}}^p \subseteq I_1^p$. Hence for $h \in \mathbb{R}_*^p$, the first alternative is ruled out, and we get

$$\forall h \in \mathbb{R}_*^p, \qquad (I_{\bar{\Delta}}^p)_\emptyset^p(h) = \emptyset.$$

Thus the inclusion in the conclusion holds trivially, and it only remains to show that the RHS of the inclusion is in $\mathcal{N}_{\ell_0}$, i.e. show that the RHS is $\emptyset$. But this is clear since by 4.1, $I_1^0 = \emptyset$, and cf. 7.2, $[h]^{p,k}_{\alpha,\beta} = \emptyset$ for $k = 0$.

This completes the proof of (a).

(b) Let $D \in \mathcal{D}_p$. Then by (a)

$$\wp_{\pi^*}[D \cap I^p_{\bar{\Delta}} \cap I^p_\pi(h)] \subseteq \wp_{\pi^*}[I^p_{\bar{\Delta}} \cap I^p_\pi(h)] \in \mathcal{N}_{\ell_k}.$$

Since $D \cap I^p_{\bar{\Delta}} \in \mathcal{D}_p$, it follows on taking the ℓ_k measure that $\lambda^p_\pi(D \cap I^p_{\bar{\Delta}}, h) = 0$. Thus (b). ■

To subsume in one formula the several cases occurring in the next theorem, and in the sequel, we shall adopt the following convention:

7.4. *Convention.* Let $k \in \mathbb{N}_+$ & $A \in \mathcal{B}_k$. Then

(a) $0 \cdot A$ shall mean a set in $\mathcal{N}_{\ell_k}$;

(b) $1 \cdot A$ shall mean a set that is ℓ_k essentially equal to A, i.e. such that the symmetric difference $1 \cdot A + A \in \mathcal{N}_{\ell_k}$.

7.5. Theorem. *Let* (i) $p \in \mathbb{N}_+$ & $k, \bar{k} \in [0, [p/2]]$, (ii) $\pi \in \Pi^p_k$ *and* $\bar{\pi} \in \Pi^p_{\bar{k}}$. *Then with the convention 7.4, we have*

(a) $\forall h \in \mathbb{R}^{p-2k}_*$, $\{I(\bar{\pi}, p)\}^p_\pi(h) = \chi_{[0,k]}(\bar{k})\chi_{2^\pi}(\bar{\pi}) \cdot \mathbb{R}^k$;

(b) *for* $\pi, \bar{\pi} \in \Pi^p_k$ & $\forall h \in \mathbb{R}^{p-2k}_*$, $\{I(\bar{\pi}, p)\}^p_\pi(h) = \delta_{\bar{\pi},\pi} \cdot \mathbb{R}^k$.

Proof. (a) *Case 1.* Let $k \in [1, [p/2]]$. There are now three subcases:

$$\bar{k} \in [0, k] \ \& \ \bar{\pi} \in 2^\pi; \qquad \bar{k} \in [0, k] \ \& \ \bar{\pi} \notin 2^\pi; \qquad \bar{k} \notin [0, k].$$

Let $h \in \mathbb{R}^{p-2k}_*$. We have to show that the equality in (a) holds in each subcase, i.e. we must show that

$$\text{(I)} \qquad \bar{k} \in [0, k] \ \& \ \bar{\pi} \in 2^\pi \implies \{I(\bar{\pi}, p)\}^p_\pi(h) = 1 \cdot \mathbb{R}^k,$$

$$\text{(II)} \qquad \bar{k} \in [0, k] \ \& \ \bar{\pi} \notin 2^\pi \implies \{I(\bar{\pi}, p)\}^p_\pi(h) = 0 \cdot \mathbb{R}^k,$$

$$\text{(III)} \qquad \bar{k} \notin [0, k] \implies \{I(\bar{\pi}, p)\}^p_\pi(h) = 0 \cdot \mathbb{R}^k.$$

We first note that since, cf. (4.7),

$$I(\pi, p) \cap I^p_\pi(h) = I^p_\pi(h) = \mathbb{R}^p \cap I^p_\pi(h),$$

therefore applying the operator $\wp_{\pi^*}$, we get

$$(1) \qquad \{I(\pi, p)\}^p_\pi(h) = (\mathbb{R}^p)^p_\pi = \mathbb{R}^k, \qquad \text{by 4.11.}$$

Proof of (I). Let $\bar{k} \leqslant k$ & $\bar{\pi} \in 2^\pi$, i.e. $\bar{\pi} \subseteq \pi$. Then, cf. (4.7),

$$I(\pi, p) := \bigcap_{\Delta \in \pi} I^p_\Delta \subseteq \bigcap_{\Delta \in \bar{\pi}} I^p_\Delta =: I(\bar{\pi}, p).$$

Hence by (1) and the monotonicity of the homomorphism in 4.11,

$$\mathbb{R}^k = \{I(\pi, p)\}^p_\pi(h) \subseteq \{I(\bar{\pi}, p)\}^p_\pi(h) \subseteq \mathbb{R}^k,$$

i.e. we have the equality in (I).

Proof of (II). Let $\bar{k} \leqslant k$ & $\bar{\pi} \notin 2^\pi$, i.e. $\bar{\pi} \not\subseteq \pi$. Then $\exists \bar{\Delta} \in \bar{\pi} \backslash \pi$. Hence by 4.3, $I(\bar{\pi}, p) := \bigcap_{\Delta \in \bar{\pi}} I^p_\Delta \subseteq I^p_{\bar{\Delta}}$, and $\bar{\Delta} \notin \pi$. Hence by the monotonicity

$$\{I(\bar{\pi}, p)\}^p_\pi(h) \subseteq (I^p_{\bar{\Delta}})^p_\pi(h) \in \mathcal{N}_{\ell_k}, \qquad \text{by 7.3}(a),$$

i.e. we have the equality in (II).

Proof of (III). Let $\bar{k} \notin [0, k]$. Then $\bar{k} < k \leqslant [p/2]$. Also $\bar{\pi}$ has more cells than π, i.e. $\exists \bar{\Delta} \in \bar{\pi}\backslash\pi$. Hence exactly as in Case II, we can conclude that the equality in (III) holds.

This finishes the proof of (a) in Case 1.

Case 2. Let $k = 0$. then $p - 2k = p$ and by (ii), $\pi = \emptyset$. Hence there are just two subcases:

$$\bar{k} = 0 \quad \& \quad \bar{\pi} = \pi = \emptyset; \quad \bar{k} \notin [0, k], \quad \text{i.e.} \quad \bar{k} \in [1, [p/2]].$$

Let $h \in \mathbb{R}^p_*$. We have to show that the equality in (a) holds in each of these two subcases, i.e. show that

$$\text{(I}'\text{)} \qquad \bar{k} = 0 \quad \& \quad \bar{\pi} = \pi = \emptyset \Longrightarrow \{I(\bar{\pi}, p)\}^p_\pi(h) = 1 \cdot \mathbb{R}^0;$$

$$\text{(II}'\text{)} \qquad \bar{k} \in [1, [p/2]] \Longrightarrow \{I(\bar{\pi}, p)\}^p_\pi(h) = 0 \cdot \mathbb{R}^0.$$

We first recall, cf. the note to 4.10, that for all $\bar{\pi} = \Pi^p_{\bar{k}}$,

$$(2) \qquad \{I(\bar{\pi}, p)\}^p_\pi(h) = \{I(\bar{\pi}, p)\}^p_\emptyset(h) = \begin{cases} \{0\} = \mathbb{R}^0, & \text{if } h \in I(\bar{\pi}, p); \\ \emptyset \in \mathcal{N}_{\ell_p} & \text{if } h \in \mathbb{R}^p \backslash I(\bar{\pi}, p). \end{cases}$$

Proof of (I$'$). Let $\bar{k} = 0$ & $\bar{\pi} = \pi = \emptyset$. Then $I(\bar{\pi}, p) = I(\emptyset, p) = \mathbb{R}^p$, by 4.3. Thus the second alternative in (2) is impossible. The first alternative prevails, i.e. we have the equality in (I$'$).

Proof of (II$'$). Let $\bar{k} \in [1, [p/2]]$. Then by 4.3, $I(\bar{\pi}, p) \subseteq I^p_1$. Now $h \in \mathbb{R}^p_* = \mathbb{R}^p \backslash I^p_1 \subseteq \mathbb{R}^p \backslash I(\bar{\pi}, p)$, i.e. the second alternative in (2) prevails. Thus $\{I(\bar{\pi}, p)\}^p_\pi(h) \in \mathcal{N}_{\ell_0}$, i.e. we have by convention 7.4, the equality in (II$'$).

This finishes the proof of (a).

(b) When $\bar{k} = k$, we have $\chi_{[0,k]}(\bar{k})\chi_{2^\pi}(\bar{\pi}) = \delta_{\bar{\pi},\pi}$ and (b) follows from (a). ■

The last theorem yields the following useful corollary on how the intersection of the set $D \in \mathcal{D}_p$ with different diagonal skeletons I^p_j shrinks the support of the canonical coefficients $\gamma^p_k(A, \cdot)$ on $\mathbb{R}^{p-2k}$:

7.6. Corollary. *Let $p \in \mathbb{N}_+$ and $k \in [0, [p/2] - 1]$. Then*
(a) $\forall j \in [k + 1, [p/2]]$ & $\forall D \in \mathcal{D}_p$, $\operatorname{supp} \gamma^p_k(D \cap I^p_j, \cdot) \in \mathcal{N}_{\ell_{p-2k}}$;
(b) in particular, $\forall D \in \mathcal{D}_p$, $\operatorname{supp} \gamma^p_k(D \cap I^p_{[p/2]}, \cdot) \in \mathcal{N}_{\ell_{p-2k}}$.

Proof. (a) To fall back on our previous notation, write $\bar{k}$ instead of j. Then $\bar{k} \in [k + 1, [p/2]]$, i.e. $\bar{k} \notin [0, k]$, and therefore by theorem 7.5(a),

$$(1) \qquad \forall h \in \mathbb{R}^{p-2k}_*, \quad \forall \pi \in \Pi^p_k \quad \& \quad \forall \bar{\pi} \in \Pi^p_{\bar{k}}, \quad \{I(\bar{\pi}, p)\}^p_\pi(h) \in \mathcal{N}_{\ell_k}.$$

Now fix $h \in \mathbb{R}^{p-2k}_*$ & $\pi \in \Pi^p_k$. Since, cf. (4.5), $I^p_{\bar{k}} = \bigcup_{\bar{\pi} \in \Pi^p_{\bar{k}}} I(\bar{\pi}, p)$, therefore appealing to the Boolean homomorphism in 4.11, we get

$$(2) \qquad (I^p_{\bar{k}})^p_\pi(h) = \bigcup_{\bar{\pi} \in \Pi^p_{\bar{k}}} \{I(\bar{\pi}, p)\}^p_\pi(h) \in \mathcal{N}_{\ell_k}, \qquad \text{by (1).}$$

Now let $D \in \mathcal{D}_p$. Then since obviously $(D \cap I^p_{\bar{k}})^p_\pi(h) \subseteq (I^p_{\bar{k}})^p_\pi(h)$, we see from (2) that $(D \cap I^p_{\bar{k}})^p_\pi(h) \in \mathcal{N}_{\ell_k}$, i.e.

$$\lambda^p_\pi(D \cap I^p_{\bar{k}})^p_\pi(h) := \ell_k[(D \cap I^p_{\bar{k}})^p_\pi(h)] = 0.$$

As this holds for all $\pi \in \Pi_k^p$, we get

$$(3) \qquad \gamma_k^p(D \cap I_{\bar{k}}^p, h) = \sum_{\pi \in \Pi_k^p} \lambda_\pi^p(D \cap I_{\bar{k}}^p, h) = 0.$$

As (3) holds for all $h \in \mathbb{R}_*^{p-2k} = \mathbb{R}^{p-2k} \backslash I_1^{p-2k}$, we see that

$$\operatorname{supp} \gamma_k^p(D \cap I_{\bar{k}}^p, \cdot) \subseteq I_1^{p-2k} \in \mathcal{N}_{\ell_{p-2k}}, \qquad \text{by (4.14)}.$$

Thus (a).

(b) just records (a) for the terminal case $j = [p/2]$. ■

This brings us to our concluding theorem, on the second absolute moment of $\xi_p(D \cap I_{p/2}^p)$ for even p:

7.7. Fundamental theorem. *Let $p \in \mathbb{N}_+$ be even. Then*

$$\forall D \in \mathcal{D}_p, \qquad \mathbb{E}_{\mathbb{P}}\{\xi_p(D \cap I_{p/2}^p)\} = |\xi_p(D \cap I_{p/2}^p)|_{\mathcal{L}_2}.$$

Proof. Let $D \in \mathcal{D}_p$ and write $C := D \cap I_{p/2}^p$ for brevity. Then by 5.7, 5.3 and 5.9,

$$|\xi_p(C)|^2 = \sum_{k=0}^{[p/2]} \Gamma_k^{pp}(C, C) = \sum_{k=0}^{(p/2)-1} \Gamma_k^{pp}(C, C) + [\mathbb{E}_{\mathbb{P}}\{\xi_p(C)\}]^2.$$

Hence we have only to show that

$$(\mathrm{I}) \qquad \forall k \in [0, (p/2) - 1], \qquad \Gamma_k^{pp}(C, C) = 0.$$

Proof of (I). Let $k \in [0, (p/2) - 1]$. Then by 5.1,

$$(1) \qquad \Gamma_k^{pp}(C, C) := \sum_{\phi \in \mathrm{Perm}(p-2k)} \int_{\mathbb{R}^{p-2k}} \gamma_k^p(C, h) \gamma_k^p(C, h^\phi) \ell_{p-2k}\,(\mathrm{d}h).$$

But by corollary 7.6(b),

$$\operatorname{supp} \gamma_k^p(C, \cdot) := \operatorname{supp} \gamma_k^p(D \cap I_{p/2}^p, \cdot) \in \mathcal{N}_{\ell_{p-2k}}.$$

Hence RHS(1) $= 0$, i.e. we have (I). ■

Thus, for even p, the second absolute moment of $\xi_p(D \cap I_{p/2}^p)$ is equal to the square of its expectation, i.e. its standard deviation vanishes. Thus

$$(1) \qquad \xi_p(D \cap I_{p/2}^p)(\cdot) = \mathbb{E}_{\mathbb{P}}\{\xi_p(D \cap I_{p/2}^p)\}, \qquad \text{a.e. } (\mathbb{P}) \text{ on } \Omega.$$

It follows from 5.9 and the fact that, cf. 7.1(b), $\forall \pi \in \Pi_{[1,p]}$, $I(\pi, p) \subseteq I_{p/2}^p$, that

$$\begin{aligned}\mathrm{RHS}(1) &= \sum_{\pi \in \Pi_{[1,p]}} \ell_{p/2}[\wp_{\pi^*}\{D \cap I_{p/2}^p \cap I(\pi, p)\}] \\ &= \sum_{\pi \in \Pi_{[1,p]}} \ell_{p/2}[\wp_{\pi^*}\{D \cap I(\pi, p)\}] = \mathbb{E}_{\mathbb{P}}\{\xi_p(D)\}.\end{aligned}$$

We thus obtain the following important corollary to the effect that *for even p, taking the intersection with the smallest diagonal skeleton $I_{p/2}^p$ reduces the random variable ξ_p to constancy*; more precisely:

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7.8. Corollary. *$\forall$ even $p \in \mathbb{N}_+$ & $\forall D \in \mathcal{D}_p$,*

$$\xi_p(D \cap I^p_{p/2})(\cdot) = \mathbb{E}_{\mathbb{P}}\{\xi_p(D)\}, \qquad \text{a.e. } (\mathbb{P}) \text{ on } \Omega.$$

8. The subspaces $\mathcal{S}_{\xi_p}$ spanned by the ξ_p

Our goal now is to study the relationship between the subspaces $\mathcal{S}_{\xi_p}$ of $\mathcal{L}_2$ spanned by the ranges of the measures ξ_p, $p \in \mathbb{N}_+$. In symbols, with the notation 1.1(f),

$$\forall p \in \mathbb{N}_+, \qquad \mathcal{S}_{\xi_p} := \mathfrak{S}\{\xi_p(D) : D \in \mathcal{D}_p\} \subseteq \mathcal{L}_2. \tag{8.1}$$

It is convenient to extend this symbolism to $p = 0$, by letting ξ_0 be the trivial measure introduced in 3.1. Then $\mathcal{S}_{\xi_0}$ is the one-dimensional subspace of $\mathcal{L}_2$ spanned by all constant-valued functions. Writing $1(\cdot)$ for the function on Ω whose value is constantly 1, we have

$$\mathcal{S}_{\xi_0} := \{c \cdot 1(\cdot) : c \in \mathbb{R}\} \subseteq \mathcal{L}_2. \tag{8.2}$$

We first assert that to determine $\mathcal{S}_{\xi_p}$ it suffices to restrict ξ_p to the pre-ring $\mathcal{P}_p$. For this we appeal to the theory of the control measure. From the local strong boundedness of ξ_p, cf. proposition 5.8, it follows that

$$\exists \mu \in \mathrm{CA}(\mathcal{D}_p, \mathbb{R}_{0+}) \ni \xi_p \ll \mu. \tag{8.3}$$

This is a theorem of Brooks (1971); for ξ_p, however, we have the specific realization $\mu = \mu_p$ given by 5.23(b). Now it is classical that if $\mu \in \mathrm{CA}(\mathcal{D}_p, \mathbb{R}_{0+})$, where $\mathcal{D} = \delta\text{-ring}(\mathcal{P})$, $\mathcal{P}$ being any pre-ring, then for all $D \in \mathcal{D}$, there exists a sequence $(P_n)_{n=1}^{\infty}$ in $\mathcal{P}$ such that $\mu(D) = \lim_{n\to\infty} \mu(P_n)$. It follows from this and (8.3) that

$$\forall D \in \mathcal{D}_p, \qquad \exists \text{ a sequence } (P_n)_{n=1}^{\infty} \text{ in } \mathcal{P}_p \ni \xi_p(D) = \lim_{n\to\infty} \xi_p(P_n).^9 \tag{8.4}$$

From (8.4) we see at once that

$$\mathcal{S}_{\xi_p} = \mathfrak{S}\{\xi_p(P) : P \in \mathcal{P}_p\}. \tag{8.5}$$

This equality is useful since it is easier to deal with $\mathcal{P}_p$ than with $\mathcal{D}_p$.

To turn to the relationship between the subspaces $\mathcal{S}_p$, we first assert that

8.6. Triviality. *$\forall p, q \in \mathbb{N}_+$, $\mathcal{S}_{\xi_p} \perp \mathcal{S}_{\xi_q} \iff p + q$ is odd.*

Proof. The implication $\Longleftarrow$ is immediate from (3.9) and (8.5). As for $\Longrightarrow$, let $p + q = 2r$ be even, and $A \in \mathcal{P}_1$ be such that $\ell_1(A) > 0$. Then $A^p \in \mathcal{P}_p$, $A^q \in \mathcal{P}_q$, and a routine computation based on the covariance formula in 3.13(a) and 6.18(c) shows that

$$(\xi_p(A^p), \xi_q(A^q)) = |\xi_r(A^r)|^2 = \{\ell_1(A)\}^r \left[r! + \sum_{k=1}^{[r/2]} \binom{r}{2}^2 (r - 2k)!(\alpha_{2k})^2 \right] > 0.$$

Thus, $\xi_p(A^p)$ is not $\perp$ to $\xi_q(A^q)$, and therefore, $\mathcal{S}_{\xi_p}$ is not $\perp$ to $\mathcal{S}_{\xi_q}$. ■

We investigate next the relationship between $\mathcal{S}_{\xi_p}$ and $\mathcal{S}_{\xi_q}$ when $q < p$ and $p + q$ is even. It turns out that $\mathcal{S}_{\xi_q} \subseteq \mathcal{S}_{\xi_p}$. To establish this far from obvious inclusion, we

[9] It is possible to prove (8.4) without appealing to (8.3), but the construction of the P_n is complicated.

first restate the important corollary 7.8 involving the diagonal skeleton $I^p_{p/2}$ in terms of $\mathcal{S}_{\xi_0}$:

(8.7) $$\forall \text{ even } p, \qquad \xi_p(D \cap I^p_{p/2}) = \mathbb{E}_{\mathbb{P}}\{\xi_p(D)\} \cdot 1(\cdot) \in \mathcal{S}_{\xi_0}.$$

In the light of (8.7), to show that $\mathcal{S}_{\xi_0} \subseteq \mathcal{S}_{\xi_p}$, for even p, we have only to select a D in $\mathcal{D}_p$ for which $\mathbb{E}_{\mathbb{P}}\{\xi_p(D)\} \neq 0$. The easiest choice is to let D be a hypercube, i.e to let $D = A^p$, where A is a (non-void) interval in $\mathcal{P}_1$. Recall that by 1.19(d)

(8.8) $$\begin{cases} \forall \text{ even } p \in \mathbb{N}_+ \ \& \ \forall D \in \mathcal{D}_1 \text{ such that } \ell_1(A) > 0, \\ \qquad\qquad \mathbb{E}_{\mathbb{P}}\{\xi_p(A^p)\} = \alpha_{p/2}\{\ell_1(A)\}^{p/2} > 0. \end{cases}$$

The set A^p serves not only in showing that $\mathcal{S}_{\xi_0} \subseteq \mathcal{S}_{\xi_p}$ for even p, but in showing more generally that $\mathcal{S}_{\xi_p} \subseteq \mathcal{S}_{\xi_{p+2k}}$:

8.9. Theorem. $\forall p, k \in \mathbb{N}_{0+}$, $\mathcal{S}_{\xi_p} \subseteq \mathcal{S}_{\xi_{p+2k}}$.

Proof. Take any A in $\mathcal{D}_1$ such that $\ell_1(A) = (1/\alpha_k)^{1/k}$. Then (8.8) tells us that

(1) $$A^{2k} \in \mathcal{D}_{2k} \quad \& \quad \mathbb{E}_{\mathbb{P}}\{\xi_{2k}(A^{2k})\} = 1.$$

Now $E_A := A^{2k} \cap I^{2k}_k \in \mathcal{D}_{2k}$, and taking $p = 2k$ in (8.7), we get, using (1),

(2) $$\xi_{2k}(E_A) = \mathbb{E}_{\mathbb{P}}\{\xi_{2k}(A^{2k})\} \cdot 1(\cdot) = 1(\cdot).$$

Now let $D \in \mathcal{D}_p$. Then by (5.13), $D \times E_A \in \mathcal{D}_{p+2k}$, and

$$\begin{aligned} \xi_p(D) = \xi_p(D) \cdot 1(\cdot) &= \xi_p(D) \cdot \xi_{2k}(E_A) && \text{by (2)} \\ &= \xi_{p+2k}(D \times E_A) \in \mathcal{S}_{\xi_{p+2k}}, && \text{by theorem 5.14.} \end{aligned}$$

It follows that $\mathcal{S}_{\xi_p} \subseteq \mathcal{S}_{\xi_{p+2k}}$. ∎

For $k \neq 0$, $\mathcal{S}_{\xi_p} \neq \mathcal{S}_{\xi_{p+2k}}$, obviously, i.e. the inclusion is proper. We may therefore sum up the relationship between the subspaces in the scheme:

(8.10) $$\begin{cases} \mathcal{S}_{\xi_0} \subset \mathcal{S}_{\xi_2} \subset \mathcal{S}_{\xi_4} \subset \cdots \subset \mathcal{S}_{\xi_{2k}} \subset \cdots \\ \mathcal{S}_{\xi_1} \subset \mathcal{S}_{\xi_3} \subset \mathcal{S}_{\xi_5} \subset \cdots \subset \mathcal{S}_{\xi_{2k+1}} \subset \cdots \\ \displaystyle\bigcup_{m=0}^{\infty} \mathcal{S}_{\xi_{2m}} \perp \bigcup_{n=0}^{\infty} \mathcal{S}_{\xi_{2n+1}}. \end{cases}$$

Let us write:

(8.11) $$\mathcal{L}^{\xi}_2 := \text{cls} \bigcup_{k=0}^{\infty} \mathcal{S}_{\xi_k}, \qquad (\mathcal{L}^{\xi}_2)^+ := \text{cls} \bigcup_{n=0}^{\infty} \mathcal{S}_{\xi_{2n}}, \qquad (\mathcal{L}^{\xi}_2)^- := \text{cls} \bigcup_{n=0}^{\infty} \mathcal{S}_{\xi_{2n+1}}.$$

Then from (8.10) it follows that

(8.12) $$\mathcal{L}^{\xi}_2 := (\mathcal{L}^{\xi}_2)^- + (\mathcal{L}^{\xi}_2)^+, \qquad (\mathcal{L}^{\xi}_2)^- \perp (\mathcal{L}^{\xi}_2)^+.$$

Taking into account the important property (8.5) of the spaces $\mathcal{S}_{\xi_p}$, it is clear that $\mathcal{L}^{\xi}_2$ is the closure of the set of all linear combinations of $1(\cdot)$ and of finite products such as $\xi(P^1) \cdot \xi(P^2) \cdots \xi(P^q)$, where $P^i \in \mathcal{D}_1$. In this, q can be any positive integer. These products include of course the powers $\xi(Q)^q$, where $Q \in \mathcal{P}_1$. We can include the unit 1 in the collection of powers by taking $q = 0 : \xi(Q)^0 = 1(\cdot)$. Thus letting

(8.13) $\mathbb{A}_{\xi} :=$ the (unitized) linear algebra over $\mathbb{R}$ spanned by $\xi_1(\mathcal{D}_1)$, i.e. by the range of ξ_1, we see that

(8.14) $$\mathcal{L}_2^\xi = \operatorname{cls} \mathbb{A}_\xi \quad \text{in} \quad \mathcal{L}_2.$$

Thus we may restate (8.12) in the form

(8.15) $$\operatorname{cls} \mathbb{A}_\xi = (\mathcal{L}_2^\xi)^- + (\mathcal{L}_2^\xi)^+, \qquad (\mathcal{L}_2^\xi)^- \perp (\mathcal{L}_2^\xi)^+.$$

The special case $\operatorname{cls} \mathbb{A}_\xi = \mathcal{L}_2$ is of interest. Recalling that $\mathcal{L}_2 = L_2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R})$, let

(8.16) $$\mathcal{A}_\xi := \sigma\text{-alg}\{\xi^{-1}(P) : P \in \mathcal{P}_1\}.$$

Then obviously $\mathcal{A}_\xi \subseteq \mathcal{A}$. We have, cf. Kakutani (1950, pp. 319–320), the following result:

8.17. Proposition. *The following conditions are equivalent:*
(α) $\operatorname{cls} \mathbb{A}_\xi = \mathcal{L}_2 = L_2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R})$,
(β) $\forall A \in \mathcal{A}, \exists B \in \mathcal{A}_\xi \ni P(A+B) = 0$, *where* $+$ *refers to the symmetric difference.*

Thus by restricting the initial σ-algebra $\mathcal{A}$ to $\mathcal{A}_\xi$ and correspondingly restricting the probability $\mathbb{P}$, we can ensure the condition 8.17(α), and affirm the orthogonal decomposition (8.15) for the entire space $\mathcal{L}_2 : \mathcal{L}_2 = (\mathcal{L}_2)^- + (\mathcal{L}_2)^+$, $\mathcal{L}_2^- \perp \mathcal{L}_2^+$.

The equality (8.7) shows that for even p, intersecting a set D in $\mathcal{D}_p$, with the diagonal skeleton $I^p_{p/2}$ has the severe effect of forcing $\xi_p(D \cap I^p_{p/2})$ to fall into $\mathcal{S}_{\xi_0}$, i.e. of turning it into a constant-valued random variable. How correspondingly severe is the effect of intersection with the other skeletons? The answer is that $\xi_p(D \cap I^p_k) \in \mathcal{S}_{\xi_{p-2k}}$, cf. 8.21 below. Interestingly, this general result can be proved by use of the special case of (8.7) for $p = 2$, namely,

(8.18) $$\forall D \in \mathcal{D}_2, \qquad \xi_2(D \cap I^2_1) \in \mathcal{S}_{\xi_0}.$$

This happens, thanks to the following simple result on the k-standard partition (6.11):

8.19. Triviality. *Let* $p \in \mathbb{N}_+$, $k \in [1, [p/2]]$ *&*

$$\pi_k = \{\{1, 2\}, \{3, 4\}, \ldots, \{2(k-1), 2k\}\} \in \varPi_{[1,2k]}.$$

Then $I(\pi_k, p) = (I^2_1)^k \times \mathbb{R}^{p-2k}$.

It is convenient to state as a separate lemma, a harder result needed in the proof of the theorem to follow:

8.20. Lemma. *Let* (i) $p \in \mathbb{N}_+$ *&* $k \in [1, [p/2]]$, (ii) $\pi \in \varPi^p_k$. *Then*

$$\forall D \in \mathcal{D}_p, \qquad \xi_p\{D \cap I(\pi, p)\} \in \mathcal{S}_{\xi_{p-2k}}.$$

Proof. We first show that it suffices to take π to be the k-standard partition π_k. For take $\phi = \phi_\pi^{-1} \in \operatorname{Perm}(p)$ as in definition 6.12, so that $\{I(\pi, p)\}^\phi = I(\pi_k, p)$, cf. 6.13($b$). Then by 6.1($a$),

$$\xi_p\{D \cap I(\pi, p)\} = \xi_p[D^\phi \cap I(\pi, p)^\phi] = \xi_p[D^\phi \cap I(\pi_k, p)].$$

It thus suffices to show that

(I) $$\forall E \in \mathcal{D}_p, \qquad \xi_p\{E \cap I(\pi_k, p)\} \in \mathcal{S}_{\xi_{p-2k}}.$$

Proof of (I). Case 1. Let $E = P = P^1 \times \cdots \times P^p$. Then

$$E = (P^1 \times P^2) \times \cdots \times (P^{2k-1} \times P^{2k}) \times \bigtimes_{i=2k+1}^{p} P^i,$$

and by triviality 8.19,

$$I(\pi_k, p) = I_1^2 \times \cdots \times I_1^2 \times \mathbb{R}^{p-2k},$$

the factor I_1^2 being repeated k times. Thus,

$$E \cap I(\pi_k, p) = \{(P^1 \times P^2) \cap I_1^2\} \times \cdots \times \{(P^{2k-1} \times P^{2k}) \cap I_1^2\} \times \bigtimes_{i=2k+1}^{p} P^i.$$

Iterated application of theorem 5.11 now yields

$$\begin{aligned}\xi_p\{E \cap I(\pi_k, p)\} &= \xi_2\{(P^1 \times P^2) \cap I_1^2\} \cdots \xi_2\{(P^{2k-1} \times P^{2k}) \cap I_1^2\} \cdot \xi_{p-2k}\left(\bigtimes_{i=2k+1}^{p} P^i\right) \\ &= c_1 1(\cdot) \cdots c_k 1(\cdot) \cdot \xi_{p-2k}\left(\bigtimes_{i=2k+1}^{p} P^i\right) \in \mathcal{S}_{\xi_{p-2k}}, \qquad \text{by (8.18).}\end{aligned}$$

Thus

(1) $$\forall P \in \mathcal{P}_p, \qquad \xi_p\{P \cap I(\pi_k, p)\} \in \mathcal{S}_{\xi_{p-2k}}.$$

Next consider $D = R \in \mathcal{R}_p$. Since $R = \bigcup_1^r P_i$, where $P_i \in \mathcal{P}_p$ are $\|$, it easily follows from the finite additivity of ξ_p that

(2) $$\forall R \in \mathcal{R}_p, \qquad \xi_p\{R \cap I(\pi_k, p)\} \in \mathcal{S}_{\xi_{p-2k}}.$$

Finally, let

$$\mathcal{F} = \{F : F \in \mathcal{D}_p \ \&\ \xi_p\{F \cap I(\pi_k, p)\} \in \mathcal{S}_{\xi_{p-2k}}\}.$$

Then by (2), $\mathcal{R}_p \subseteq \mathcal{F} \subseteq \mathcal{D}_p$, whence $\mathcal{D}_p = \delta\text{-mon}(\mathcal{F})$. But it is easily seen that $\mathcal{F}$ is itself a δ-monotone class. For let $\forall k \in \mathbb{N}_+$, $F_k \in \mathcal{D}$ & $F_k \downarrow E$. Then $E \in \mathcal{D}_p$ and since each $\xi_p(F_k) \in \mathcal{S}_{\xi_{p-2k}}$, it follows from the countable additive of ξ_p that

$$\xi_p(E) = \lim_{n\to\infty} \xi_p(F_k) \in \mathcal{S}_{\xi_{p-2k}}.$$

Thus, $E \in \mathcal{F}$. The same argument shows that

$$\forall k \in \mathbb{N}_+, \qquad F_k \in \mathcal{F} \quad \& \quad F_k \uparrow E \subseteq F \in \mathcal{F} \Longrightarrow E \in \mathcal{F}.$$

Thus, $\mathcal{D}_p = \delta\text{-mon}(\mathcal{F}) = \mathcal{F}$, i.e. $\forall D \in \mathcal{D}_p$, $\xi_p\{D \cap I(\pi_k, p)\} \in \mathcal{S}_{\xi_{p-2k}}$. This proves (I). ∎

8.21. Theorem. *Let $p \in \mathbb{N}_+$ & $k \in [1, [p/2]]$. Then $\forall D \in \mathcal{D}_p$, $\xi_p(D \cap I_k^p) \in \mathcal{S}_{\xi_{p-2k}}$.*

Proof. Let $D \in \mathcal{D}_p$. Then

$$D \cap I_k^p = \bigcup_{\pi \in \Pi_k^p} \{D \cap I(\pi, p)\}.$$

There are, cf. (1.17), $r := \binom{q}{2k}\alpha_{2k}$ terms in this union, which we may denumerate in any order by denumerating the partitions π as $\pi_1, \pi_2, \ldots, \pi_r$. Then, writing $\forall i \in [1, r]$, $D_{\pi_i} := D \cap I(\pi_i, p)$, we have $D \cap I_k^p = \bigcup_{i=1}^r D_{\pi_i}$, whence by the inclusion-exclusion principle

$$\begin{aligned}\xi_p(D \cap I_k^p) &= \sum_{i=1}^{r} \xi_p(D_{\pi_i}) - \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} \xi_p(D_{\pi_i} \cap D_{\pi_j}) \\ &\quad + \cdots + (-1)^{r+1} \xi_p(D_{\pi_1} \cap D_{\pi_2} \cdots \cap D_{\pi_r})\end{aligned}$$

(1) $$= S_1 - S_2 + \cdots + (-1)^{r+1} S_r, \quad \text{say.}$$

Now by lemma 8.20, $\forall i \in [1,r]$, $\xi_p(D_{\pi_i}) := \xi[D \cap I(\pi_i, p)] \in \mathcal{S}_{\xi_{p-2k}}$. Hence $S_1 \in \mathcal{S}_{\xi_{p-2k}}$. Next consider the generic term in S_2, namely,

$$\xi(D_{\pi_i} \cap D_{\pi_j}) = \xi_p\{D \cap I(\pi_i, p) \cap I(\pi_j, p)\}. \tag{2}$$

Since $D \cap I(\pi_i, p) \in \mathcal{D}_p$, it follows from lemma 8.20 that RHS(2) $\in \mathcal{S}_{\xi_{p-2k}}$. Thus, $\xi(D_{\pi_i} \cap D_{\pi_j}) \in \mathcal{S}_{\xi_{p-2k}}$, whence

$$S_2 := \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} \xi(D_{\pi_i} \cap D_{\pi_j}) \in \mathcal{S}_{\xi_{p-2k}}.$$

Proceeding in this way, we can show that $S_3, S_4, \ldots, S_r \in \mathcal{S}_{\xi_{p-2k}}$. The result now follows from (1). ■

9. Concordance of the orthogonal and Lebesgue decompositions $\eta_p + \zeta_p$ of ξ_p

Let $\mathcal{L}_0$ be a (closed linear) subspace of $\mathcal{L}_2$. Denoting by $\mathrm{proj}(x|\mathcal{L}_0)$ the orthogonal projection of a vector $x \in \mathcal{L}_2$ on $\mathcal{L}_0$, we now introduce the measures obtained by the projection of ξ_p onto $\mathcal{S}_{\xi_q}$ and onto $\mathcal{S}_{\xi_q}^{\perp}$:

9.1. *Definition.* Let $p \in \mathbb{N}_+$ & $q \in \mathbb{N}_{+0}$ be such that $p+q$ be even and $q < p$. Then, $\forall D \in \mathcal{D}_p$,

$$\eta_{p,q}(D) := \mathrm{proj}(\xi_p(D)|\mathcal{S}_{\xi_q}^{\perp}), \qquad \zeta_{p,q}(D) := \mathrm{proj}(\xi_p(D)|\mathcal{S}_{\xi_q}).$$

It follows of course that

$$\begin{cases} \forall p, q \text{ as in definition 9.1}, \quad \eta_{p,q}, \zeta_{p,q} \in \mathrm{CA}(\mathcal{D}_p, \mathcal{L}_2) \quad \& \\ \forall D \in \mathcal{D}_p, \quad \xi_p(D) = \eta_{p,q}(D) + \zeta_{p,q}(D), \quad \eta_{p,q}(D) \perp \zeta_{p,q}(D). \end{cases} \tag{9.2}$$

From this we readily get the following result:

9.3. Triviality. *Let p, q be as in definition 9.1,*
(a) $\mathcal{S}_{\xi_p} = \mathcal{S}_{\xi_q} + \mathcal{S}_{\eta_{p,q}}$, $\mathcal{S}_{\xi_q} \perp \mathcal{S}_{\eta_{p,q}}$;
(b) $\mathcal{S}_{\eta_{p,q}} = \mathcal{S}_{\xi_p} \cap \mathcal{S}_{\xi_q}^{\perp}$.

Proof. (*a*) The orthogonality is obvious from the definitions of $\eta_{p,q}$ and $\zeta_{p,q}$, as is the inclusion

$$\mathcal{S}_{\xi_p} \subseteq \mathcal{S}_{\xi_q} + \mathcal{S}_{\eta_{p,q}}. \tag{1}$$

Next, by the crucial inclusions (8.10),

$$\eta_{p,q}(D) = \xi_p(D) - \zeta_{p,q}(D) \in \mathcal{S}_{\xi_p} + \mathcal{S}_{\xi_q} \subseteq \mathcal{S}_{\xi_p},$$

whence $\mathcal{S}_{\eta_{p,q}} \subseteq \mathcal{S}_{\xi_p}$ and therefore certainly

$$\mathcal{S}_{\xi_q} + \mathcal{S}_{\eta_{p,q}} \subseteq \mathcal{S}_{\xi_p}. \tag{2}$$

By (1) and (2) we have the equality in (*a*). Hence (*a*).
(*b*) is just a restatement of (*a*). ■

The next theorem asserts that the projection $\zeta_{p,q}(D)$ can be had by intersecting D with the $(p-q)/2$th diagonal skeleton of $\mathbb{R}^p$, roughly speaking. More precisely:

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9.4. Theorem. *Let $p \in \mathbb{N}_+$ & $j \in [1,[p/2]]$. Then, $\forall D \in \mathcal{D}_p$,*

$$\zeta_{p,p-2j}(D) = \xi_p(D \cap I_j^p), \qquad \eta_{p,p-2j}(D) = \xi_p(D \backslash I_j^p);$$

in particular (for $j = [p/2]$)
for even p, $\zeta_{p,0}(D) = \xi_p(D \cap I_{p/2}^p)$, $\eta_{p,0}(D) = \xi_p(D \backslash I_{p/2}^p)$,
for odd p, $\zeta_{p,1}(D) = \xi_p(D \cap I_{[p/2]-1}^p)$, $\eta_{p,1}(D) = \xi_p(D \backslash I_{[p/2]-1}^p)$.

Proof. Let $D \in \mathcal{D}_p$. Since by theorem 8.21, $\xi_p(D \cap I_j^p) \in \mathcal{S}_{\xi_{p-2j}}$, to prove the first equality we have only to show that

(I) $$\xi_p(D) - \xi_p(D \cap I_j^p) \perp \mathcal{S}_{\xi_{p-2j}}.$$

Proof of (I). Let $E \in \mathcal{D}_{p-2j}$. Then by the covariance equality 5.3,

$$(\xi_p(D) - \xi_p(D \cap I_j^p), \xi_{p-2j}(E)) = (\xi_p(D \backslash I_j^p), \xi_{p-2j}(E))$$

(1) $$= \sum_{k=0}^{[(p-2j)/2]} \Gamma_k^{p,p-2j}(D \backslash I_j^p, E).$$

Writing $\bar{D} = D \backslash I_j^p$, we have, cf. 5.1

(2) $$\Gamma_0^{p,p-2j}(\bar{D}, E) := \sum_{\phi \in \mathrm{Perm}(p-2j)} \int_{\mathbb{R}^{p-2j}} \gamma_j^p(\bar{D}, h) \chi_E(h^\phi) \ell_{p-2j}(\mathrm{d}h),$$

and $\forall k \in [1, [(p-2j)/2]]$,

(3) $$\Gamma_k^{p,p-2j}(\bar{D}, E) := \sum_{\phi \in \mathrm{Perm}(p-2j-2k)} \int_{\mathbb{R}^{p-2j-2k}} \gamma_{j+k}^p(\bar{D}, h) \gamma_k^{p-2j}(E, h^\phi) \ell_{p-2j-2k}(\mathrm{d}h).$$

Now grant momentarily that

(A) $$\forall k \in [0, [(p-2j)/2]], \quad \forall \pi \in \Pi_{j+k}^p \quad \& \quad \forall h \in \mathbb{R}^{p-2(j+k)}, \quad \lambda_\pi^p(\bar{D}, h) = 0.$$

Then, $\forall k \in [0, [(p-2j)/2]]$,

$$\gamma_{j+k}^p(\bar{D}, h) := \sum_{\pi \in \Pi_{j+k}^p} \lambda_\pi^p(\bar{D}, h) = 0,$$

whence by (2) and (3) for each $k \in [0, [(p-2j)/2]$, $\Gamma_k^{p,p-2j}(\bar{D}, h) = 0$, and by (1), $\xi_p(D) - \xi_p(D \cap I_j^p) \perp \xi_{p-2j}(E)$. As this holds for any $E \in \mathcal{D}_{p-2j}$, we have (I).

It remains to justify (A).

Proof of (A). Let k, π, h be as in (A). Then

(4) $$\lambda_\pi^p(\bar{D}, h) := \ell_{j+k}[\wp_{\pi^*}\{(D \backslash I_j^p) \cap I_\pi^p(h)\}] = \ell_{j+k}[\wp_{\pi^*}\{D \cap (I_\pi^p(h) \backslash I_j^p)\}].$$

But by (4.7) and 7.1(a), $I_\pi^p(h) \subseteq I(\pi, p) \subseteq I_{j+k}^p \subseteq I_j^p$, i.e. $I_\pi^p(h) \backslash I_j^p = \emptyset$. Hence by (4), $\lambda_\pi^p(\bar{D}, h) = 0$. Thus (A).

This establishes (I) and with it the first equality in 9.4. The second obviously follows from this and the equality in (9.2). ∎

In exact analogy with 5.22, we deduce as a corollary of 9.4:

9.5. Corollary. *Let $p \in \mathbb{N}_+$ & $j \in [1,[p/2]]$. Then $\forall x' \in (\mathcal{L}_2)'$ & $\forall A \in \mathcal{B}_p$,*

$$|x' \circ \eta_{p,p-2j}|(A) = |x' \circ \xi_p|(A \backslash I_j^p), \qquad |x' \circ \zeta_{p,p-2j}|(A) = |x' \circ \zeta_p|(A \cap I_j^p),$$

$$|x' \circ \xi_p|(A) = |x' \circ \eta_{p,p-2j}|(A) + |x' \circ \zeta_{p,p-2j}|(A).$$

The two-step projections $\eta_{p,p-2}$ and $\zeta_{p,p-2}$, obtained by taking $j = 1$, are the really important ones and we give them a special notation: $\eta_p := \eta_{p,p-2}$, $\zeta_p := \zeta_{p,p-2}$. More fully,

9.6. *Definition.* For $p \in \mathbb{N}_+$, $p \geqslant 2$, we define, $\forall D \in \mathcal{D}_p$,

$$\eta_p(D) := \mathrm{proj}(\xi_p(D)|\mathcal{S}^{\perp}_{\xi_{p-2}}), \qquad \zeta_p(D) := \mathrm{proj}(\xi_p(D)|\mathcal{S}_{\xi_{p-2}}).$$

For completeness we set $\eta_1 := \xi_1$ and $\zeta_1 := 0$, $\eta_0 := \xi_0$.

As a special case of (9.2), we have

$$\text{(9.7)} \qquad \begin{cases} \forall p \in \mathbb{N}_+, \quad \eta_p, \zeta_p \in \mathrm{CA}(\mathcal{D}_p, \mathcal{L}_2) \ \& \\ \forall D \in \mathcal{D}, \quad \xi_p(D) = \eta_p(D) + \zeta_p(D), \quad \eta_p(D) \perp \zeta_p(D). \end{cases}$$

Also, triviality 9.3 yields as a special case the following orthogonal decomposition for the subspaces $\mathcal{S}_{\xi_p}$:

9.8. Triviality. *Let $p, q \in \mathbb{N}_+$ & $2 \leqslant q < p$. Then*

(*a*) $\mathcal{S}_{\xi_p} = \mathcal{S}_{\eta_p} + \mathcal{S}_{\xi_{p-2}}$, $\mathcal{S}_{\eta_p} \perp \mathcal{S}_{\xi_{p-2}}$;

(*b*) $\mathcal{S}_{\eta_p} = \mathcal{S}_{\xi_p} \cap \mathcal{S}^{\perp}_{\xi_{p-2}}$;

(*c*) $\mathcal{S}_{\eta_p} \perp \mathcal{S}_{\eta_q}$, *even for $q = 0$ and 1.*

By iteration of the decomposition 9.8(*a*), we obtain the orthogonal decompositions:

$$\text{(9.9)} \qquad \begin{cases} \forall \text{ odd } p > 1, \quad \mathcal{S}_{\xi_p} = \mathcal{S}_{\eta_p} + \mathcal{S}_{\eta_{p-2}} + \cdots + \mathcal{S}_{\eta_3} + \mathcal{S}_{\eta_1}, \\ \forall \text{ even } p > 0, \quad \mathcal{S}_{\xi_p} = \mathcal{S}_{\eta_p} + \mathcal{S}_{\eta_{p-2}} + \cdots + \mathcal{S}_{\eta_2} + \mathcal{S}_{\eta_0}, \\ \forall p, q \in \mathbb{N}_+ \ni p + q \text{ is even } \& \ q < p, \\ \mathcal{S}_{\xi_p} = \displaystyle\sum_{j=1}^{[(p-q)/2]} \mathcal{S}_{\eta_{q+2j}} + \mathcal{S}_{\xi_q}, \quad \sum_{j=1}^{[(p-q)/2]} \mathcal{S}_{\eta_{q+2j}} \perp \mathcal{S}_{\xi_q}. \end{cases}$$

It follows at once from the first two equalities in (9.9) that

$$\text{(9.10)} \qquad \forall p \in \mathbb{N}_+ \ \& \ \forall D \in \mathcal{D}_p, \qquad \xi_p(D) = \sum_{k=0}^{[p/2]} \mathrm{proj}(\xi_p(D)|\mathcal{S}_{\eta_{p-2k}}).$$

To turn to the connection between η_p, ζ_p and the Lebesgue components ξ^a_p, ξ^b_p, note that on taking $j = 1$ in 9.4, we get

$$\text{(9.11)} \qquad \forall D \in \mathcal{D}_p, \qquad \zeta_p(D) = \xi_p(D \cap I^p_1), \qquad \eta_p(D) = \xi_p(D \backslash I^p_1).$$

On comparing this with the definitions of ξ^a_p, ξ^b_p in (5.16), and recalling the Lebesgue decomposition theorem 5.18, we get at once the following useful result:

9.12. Concordance theorem. *Let $p \in \mathbb{N}_+$. Then there is a concordance between* *the two-step orthogonal decomposition of* ξ_p*, and the Lebesgue decomposition of* ξ_p *with respect to* ℓ_p*. More precisely,*

$$\forall D \in \mathcal{D}_p, \qquad \zeta_p(D) = \xi_p^b(D) \quad \& \quad \eta_p(D) = \xi_p^a(D).$$

Moreover, η_p *and* ℓ_p *are equivalent, i.e.* $\mathcal{N}_{\eta_p} = \mathcal{N}_{\ell_p}$.

In the light of this theorem we shall abandon the notation ξ_p^a, ξ_p^b in favour of the shorter η_p, ζ_p. For ready reference in the sequel we restate the results 5.17(a)–(d), 5.18(b), (6.3), 5.22 on ξ_p^a, ξ_p^b in terms of η_p, ζ_p:

9.13. Proposition. (Summary) *Let* $p \in \mathbb{N}_+$*. Then*

(a) $\forall D, E \in \mathcal{D}_p,\ (\eta_p(D), \eta_p(E)) = \sum_{\phi\in\mathrm{Perm}(p)} \ell_p(D \cap E^\phi)$, $\mathbb{E}_{\mathbb{P}}\{\eta_p(\cdot)\} = 0$ *on* $\mathcal{D}_p$;

(b) $\forall D \in \mathcal{D}_p,\ \sqrt{\ell_p(D)} \leqslant q_{\eta_p}(D) = |\eta_p(D)| = s_{\eta_p}(D) \leqslant \sqrt{p!}\sqrt{\ell_p(D)}$;

(c) $\forall A \in \mathcal{B}_p,\ \sqrt{|\ell_p|(A)} \leqslant q_{\eta_p}(A) = s_{\eta_p}(A) \leqslant \sqrt{p!}\sqrt{|\ell_p|(A)}$;

(d) η_p & ℓ_p *are equivalent:* $\mathcal{N}_{\eta_p} = \mathcal{N}_{\ell_p}$, & $\ell_p \perp\!\!\!\perp \zeta_p$;

(e) $\forall D, E \in \mathcal{D}_p$,

$$(\zeta_p(D), \zeta_p(E)) = \sum_{k=1}^{[p/2]} \Gamma_k^{pp}(D, E);$$

(f) $\forall \phi \in \mathrm{Perm}(p)$ & $\forall D \in \mathcal{D}_p,\ \eta_p(D^\phi) = \eta_p(D)$, $\zeta_p(D^\phi) = \zeta_p(D)$;

(g) $\forall \phi \in \mathrm{Perm}(p)$, $\forall x' \in (\mathcal{L}_2)'$ & $\forall A \in \mathcal{B}_p$,

$$|x' \circ \eta_p|(A^\phi) = |x' \circ \eta_p|(A), \qquad |x' \circ \zeta_p|(A^\phi) = |x' \circ \zeta_p|(A);$$

$$|x' \circ \xi_p|(A) = |x' \circ \eta_p|(A) + |x' \circ \zeta_p|(A);$$

(h) $\forall A \in \mathcal{B}_p,\ \frac{1}{2}\{s_{\eta_p}(A) + s_{\zeta_p}(A)\} \leqslant s_{\xi_p}(A) \leqslant s_{\eta_p}(A) + s_{\zeta_p}(A)$.

9.14. *Remarks.* A comparison of the formulae in 9.8(c), 9.13(a) and 5.3 shows that the covariance structure of η_p is considerably simpler than that of ξ_p. Indeed when restricted to the δ-ring $\mathcal{D}_p^{\mathrm{sym}}$ of symmetric subsets of $\mathcal{D}_p$, the η_p are *biorthogonally scattered*:

$$\forall D \in \mathcal{D}_p^{\mathrm{sym}} \quad \& \quad \forall E \in \mathcal{D}_q^{\mathrm{sym}}, \qquad (\eta_p(D), \eta_q(E)) = p!\ell_p(D \cap E) \cdot \delta_{pq}.$$

Thus, $\mathrm{Rstr.}_{\mathcal{D}_p^{\mathrm{sym}}}\, \eta_p \in \mathrm{CAOS}(\mathcal{D}_p^{\mathrm{sym}}, \mathcal{L}_2)$.

An expression for $\mathrm{proj}(\xi_p(D)|\mathcal{S}_{\eta_q})$ in terms of $\xi_p(\cdot)$, where $q < p$ and $p - q$ is even, involving diagonal skeletons as in theorem 9.4, emerges as a simple corollary of the latter. It is convenient to introduce the notation

(9.15) $\forall p \in \mathbb{N}_+, \quad \forall k \in [0, [p/2]] \quad \& \quad \forall D \in \mathcal{D}_p, \quad \xi_{p,k}(D) := \mathrm{proj}(\xi_p(D)|\mathcal{S}_{\eta_{p-2k}})$.

Note. Obviously, $\xi_{p,0}(D) = \eta_p(D)$, and

$$\xi_{p,[p/2]}(D) := \begin{cases} \mathrm{proj}(\xi_p(D)|\mathcal{S}_{\xi_1}) = \zeta_{p,1}(D), & p \text{ odd}, \\ \mathrm{proj}(\xi_p(D)|\mathcal{S}_{\xi_0}) = \zeta_{p,0}(D), & p \text{ even}. \end{cases}$$

With this notation, we have:

9.16. Corollary. *Let* $p \in \mathbb{N}_+$ & $k \in [0, [p/2]]$*. Then*

(a) $\forall D \in \mathcal{D}_p,\ \xi_{p,k}(D) = \xi_p\{D \cap (I_k^p \backslash I_{k+1}^p)\}$, *where* $I_{[p/2]+1}^p := \emptyset$;

(b) $\forall x' \in (\mathcal{L}_2)'$ & $\forall A \in \mathcal{B}_p,\ |x' \circ \xi_{p,k}|(A) = |x' \circ \xi_p|\{A \cap (I_k^p \backslash I_{k+1}^p)\}$;

(c) $\forall \phi \in \mathrm{Perm}(p)$, $\forall x' \in (\mathcal{L}_2)'$, $\forall D \in \mathcal{D}_p$ & $\forall A \in \mathcal{B}_p$,

$$\xi_{p,k}(D^\phi) = \xi_{p,k}(D) \quad \& \quad |x' \circ \xi_{p,k}|(A^\phi) = |x' \circ \xi_{p,k}|(A).$$

Proof. (*a*) Let $D \in \mathcal{D}_p$ & $0 \leqslant k \leqslant [p/2]-1$. Then since by (9.15), $\xi_{p,k}(D) = \operatorname{proj}(\xi_p(D)|\mathcal{S}_{\eta_{p-2k}})$ and by 9.8(*b*), $\mathcal{S}_{\eta_{p-2k}} = \mathcal{S}_{\xi_{p-2k}} \cap \mathcal{S}^{\perp}_{\xi p-2k-2}$, it follows, cf. 9.1, that

$$\begin{aligned} \xi_{p,k}(D) &= \zeta_{p,p-2k}(D) - \zeta_{p,p+2k-2}(D) \\ &= \xi_p(D \cap I^p_k) - \xi_p(D \cap I^p_{k+1}) \qquad \text{by theorem 9.4} \\ &= \xi_p\{D \cap (I^p_k \backslash I^p_{k+1})\}, \qquad \text{since } I^p_{k+1} \subseteq I^p_k. \end{aligned}$$

But the last equality also holds for $k = [p/2]$, for then $I^p_{k+1} := \emptyset$, and we know from 9.4 that

$$\xi_{p,[p/2]}(D) = \left\{ \begin{array}{c} \zeta_{p,0}(D) \\ \zeta_{p,1}(D) \end{array} \right\} = \xi_p(D \cap I^p_{[p/2]}).$$

Thus (*a*).

(*b*) follows on applying the classical triviality that if μ_0 is defined on $\mathcal{D}_p$ by $\mu_0(\cdot) = \mu(\cdot \cap B)$, where $B \in \mathcal{B}_p$, then $|\mu_0|(\cdot) = |\mu|(\cdot \cap B)$ on $\mathcal{B}_p$.

(*c*) follows at once from (*a*) and (*b*) by virtue of the symmetry of $I^p_k \backslash I^p_{k+1}$, cf. 7.1(*c*) ■

Theorem 9.4 and corollary 9.16(*a*) show that there is a kind of isomorphism between the calculus of projections of $\xi_p(D)$ on the $\mathcal{S}_{\xi_q}$ and $\mathcal{S}^{\perp}_{\xi_q}$, and the set-theoretic calculus of diagonal skeletons.

To turn to the expectations of the measures we have been considering, much light is shed by the easily proved result:

(9.17) $$\forall x \in \mathcal{L}_2, \qquad \operatorname{proj}(x|\mathcal{S}_{\xi_0}) = \mathbb{E}_{\mathbb{P}}(x) \cdot 1(\cdot).$$

We have the following proposition:

9.18. Proposition. *Let $p \in \mathbb{N}_+$ & $D \in \mathcal{D}_p$. Then*

(*a*) *for odd p, $\mathbb{E}_{\mathbb{P}}\{\xi_p(D)\} = 0$, and for even p,*

$$\mathbb{E}_{\mathbb{P}}\{\xi_p(D)\} = |\xi_p\{D \cap I^p_{p/2}\}| = \sum_{\pi \in \Pi_{[1,p]}} \ell_{p/2}\{D \cap I(\pi,p)\};$$

(*b*) *for $1 \leqslant q < p$ such that $p-q$ is even,*

$$\mathbb{E}_{\mathbb{P}}\{\eta_{p,q}(D)\} = 0 \quad \& \quad \mathbb{E}_{\mathbb{P}}\{\zeta_{p,q}(D)\} = \mathbb{E}_{\mathbb{P}}\{\xi_p(D)\}.$$

Proof. (*a*) For odd p, see 5.9(*b*). For even p, the two equalities just repeat theorem 7.7 and 5.9(*a*).

(*b*) For odd p, $\mathbb{E}_{\mathbb{P}}\{\eta_{p,q}(D)\} = \mathbb{E}_{\mathbb{P}}\{\xi(D \backslash I^p_{(p-q)/2})\} = 0$, by 9.4 and 5.9(*b*). For even p, since $\mathcal{S}_{\xi_0} \subseteq \mathcal{S}_{\xi_q} \perp \mathcal{S}^{\perp}_{\xi_q}$ and $\eta_{p,q}(D) := \operatorname{proj}(\xi_p(D)|\mathcal{S}^{\perp}_{\xi_q}) \perp \mathcal{S}_{\xi_0}$, it is clear from (9.17) that $\mathbb{E}_{\mathbb{P}}\{\eta_{p,q}(D)\} = 0$. Since $\zeta_{p,q} = \xi_p - \eta_{p,q}$, it follows that $\mathbb{E}_{\mathbb{P}}\{\zeta_{p,q}(D)\} = \mathbb{E}_{\mathbb{P}}\{\xi_p(D)\}$. Thus (*b*). ■

Taking $q = p-2$ in 9.18(*b*), we recover the second equation in 9.13(*a*):

(9.19) $$\forall p \in \mathbb{N}_+ \quad \& \quad \forall D \in \mathcal{D}_p, \quad \mathbb{E}_{\mathbb{P}}\{\eta_p(D)\} = 0 \quad \& \quad \mathbb{E}_{\mathbb{P}}\{\zeta_p(D)\} = \mathbb{E}_{\mathbb{P}}\{\xi_p(D)\}.$$

Finally, we note that the orthogonality of the decompositions in (9.9) allows us to infer the following orthogonal decomposition of the subspace $\mathcal{L}^{\xi}_2$ of $\mathcal{L}_2$ defined in (8.11):

(9.20) $$\mathcal{L}^{\xi}_2 = \sum_{p=0}^{\infty} \mathcal{S}_{\eta_p}, \quad (\mathcal{L}^{\xi}_2)^- = \sum_{k=0}^{\infty} \mathcal{S}_{\eta_{2k+1}}, \quad (\mathcal{L}^{\xi}_2)^+ = \sum_{k=0}^{\infty} \mathcal{S}_{\eta_{2k}}.$$

Part II. Chaotic integration

10. Integrability and integration with respect to the measure η_p

Because of the greater simplicity of the covariance structure of η_p in comparison with that of ξ_p, cf. 9.14, we shall first attend to integrability and integration with respect to η_p. Later (§ 13) we shall show that integration with respect to ξ_p is reducible to integration with respect to η_p. The rudiments, however, which apply to both measures, are dealt with in Appendix A under the neutral notation

$$\rho \in \mathrm{CA}(\mathcal{D}, \mathcal{H}),$$

where $\mathcal{D}$ is any δ-ring over a space Λ, and $\mathcal{H}$ any Hilbert space. There the fundamental classes $S(\mathcal{D}, \mathbb{R})$, $\mathcal{P}_{1,\rho}$ of $\mathcal{D}$-simple and ρ-integrable functions, respectively, are defined, as is the operator $\mathbb{E}_\rho$ of integration with respect to ρ, cf. (A.12), (A.10). The reader is requested to consult this appendix.

To turn to results specific to the measure η_p, we have first:

10.1. Lemma. *$\forall f, g \in S(\mathcal{D}_p, \mathbb{R})$, cf. (A.4),*

$$(*) \qquad (\mathbb{E}_{\eta_p}(f), \mathbb{E}_{\eta_p}(g)) = p! \int_{\mathbb{R}^p} \tilde{f}(t)\tilde{g}(t)\ell_p\,(\mathrm{d}t),$$

where $\tilde{f}$, $\tilde{g}$ are the symmetrizations of f and g.

Proof. Let $f = \sum_{k=1}^r a_k \chi_{D_k}$, $g = \sum_{k=1}^r b_k \chi_{E_k}$ & $D_k, E_k \in \mathcal{D}_p$. Then using the definition (A.25), of $\mathbb{E}_{\eta_p}(f)$ for simple f, and 9.13(a), it is easily checked that the LHS in $(*)$ is equal to

$$(1) \qquad \sum_{j=1}^r \sum_{k=1}^r a_j b_k \sum_{\phi \in \mathrm{Perm}(p)} \ell_p(D_j \cap E_k^{\phi^{-1}}).$$

Next, symmetrization ˜ being linear, cf. 1.46, the RHS in $(*)$ is equal to

$$(2) \qquad p! \sum_{j=1}^r \sum_{k=1}^r a_j b_k (\tilde{\chi}_{D_j}, \tilde{\chi}_{E_k})_{2,\ell_p}.$$

But since ˜ is in fact an orthogonal projection, cf. 6.19, the last inner product equals

$$\begin{aligned}(\tilde{\chi}_{D_j}, \tilde{\chi}_{E_k})_{2,\ell_p} = (\chi_{D_j}, \tilde{\chi}_{E_k})_{2,\ell_p} &= \frac{1}{p!} \sum_{\phi \in \mathrm{Perm}(p)} (\chi_{D_j}, \chi_{E_k^{\phi^{-1}}})_{2,\ell_p} \\ &= \frac{1}{p!} \sum_{\phi \in \mathrm{Perm}(p)} \ell_p(D_j \cap E_k^{\phi^{-1}}).\end{aligned}$$

Thus the RHS in $(*)$ also equals (1). ■

We next assert:

10.2. Lemma. *Let $f \in \mathcal{P}_{1,\eta_p}$. Then*
(a) *$f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$, $\tilde{f} \in \mathcal{P}_{1,\eta_p}$ & $\tilde{f} \in L_2(\mathbb{R}^p)$;*
(b) *$\forall s \in S(\mathcal{D}_p, \mathbb{R})$, $\sqrt{p!}\,|\tilde{f} - \tilde{s}|_{2,\ell_p} \leqslant |f - s|_{1,\eta_p}$.*

Proof. (a) The first statement is a part of the definition of $\mathcal{P}_{1,\eta_p}$, cf. (A.17) and

(A.10), and that $\tilde{f} \in \mathcal{P}_{1,\eta_p}$ is clear, since η_p is a permutation-invariant, cf. 9.13(f) and A.35(b).

Next, by (A.14) and (A.26), there exists $(s_n)_{n=1}^{\infty}$ in $S(\mathcal{D}_p, \mathbb{F})$ such that

$$(1) \qquad |\mathbb{E}_{\eta_p}(f - s_n)| \leqslant |(f - s_n)|_{1,\eta_p} \to 0, \quad \text{as} \quad n \to \infty.$$

Thus, $\forall m, n \in \mathbb{N}_+$,

$$|\mathbb{E}_{\eta_p}(s_m - s_n)| \leqslant |s_m - s_n|_{1,\eta_p} \to 0, \quad \text{as} \quad m, n \to \infty,$$

i.e. by the last lemma (with $f = g = |s_m - s_n|$),

$$(2) \qquad \sqrt{p!}\, |\tilde{s}_m - \tilde{s}_n|_{2,\ell_p} \leqslant |s_m - s_n|_{1,\eta_p} \to 0, \quad \text{as} \quad m, n \to \infty.$$

It follows from (2) that

$$(3) \qquad \exists g \in L_2(\mathbb{R}^p) \ni |\tilde{s}_n - g|_{2,\ell_p} \to 0, \quad \text{as} \quad n \to \infty.$$

We now appeal to the subsequence principle A.15, and obtain a subsequence $(s_{n_k})_{k=1}^{\infty}$ and a set $N \in \mathcal{N}_{\eta_p} = \mathcal{N}_{\ell_p}$, cf. 9.13($d$), such that

$$\forall t \in \mathbb{R}^p \backslash N, \qquad \lim_{k \to \infty} s_{n_k}(t) = f(t) \quad \& \quad \lim_{k \to \infty} \tilde{s}_{n_k}(t) = g(t).$$

It follows that $\tilde{f} = g$, a.e. ℓ_p and a.e. η_p. Hence by (3), $\tilde{f} \in L_2(\mathbb{R}^p)$. Thus ($a$).

(b) Let $s \in S(\mathcal{D}_p, \mathbb{R})$. Then exactly as with (2), we have

$$(4) \qquad \forall m \in \mathbb{N}_+, \qquad \sqrt{p!}\, |\tilde{s}_m - \tilde{s}|_{2,\ell_p} \leqslant |s_m - s|_{1,\eta_p}.$$

Letting $m \to \infty$, we conclude from (3) and (1) that

$$\sqrt{p!}\, |g - \tilde{s}|_{2,\ell_p} \leqslant |f - s|_{1,\eta_p}.$$

Since $g = \tilde{f}$, we have (b). ∎

10.3. Theorem. *Let* $p \in \mathbb{N}_+$ *&* $f, g \in \mathcal{P}_{1,\eta_p}$. *Then*

(a) $(\mathbb{E}_{\eta_p}(f), \mathbb{E}_{\eta_p}(g))_{\mathcal{L}_2} = p!(\tilde{f}, \tilde{g})_{2,\ell_p}$;

(b) $|\mathbb{E}_{\eta_p}(f)|_{\mathcal{L}_2} = \sqrt{p!}\, |\tilde{f}|_{2,\ell_p} \leqslant \sqrt{p!}\, |f|_{2,\ell_p}$, *for symmetric* f, $|\mathbb{E}_{\eta_p}(f)|_{\mathcal{L}_2} = \sqrt{p!}\, |f|_{2,\ell_p}$;

(c) $\mathbb{E}_{\eta_p}(f) = \mathbb{E}_{\eta_p}(\tilde{f})$;

(d) $\mathbb{E}_{\mathbb{P}}\{\mathbb{E}_{\eta_p}(f)\} = 0$.

Proof. (a) By (A.14), there exists functions $s_n^1, s_n^2 \in S(\mathcal{D}_p, \mathbb{F})$ such that

$$(1) \qquad |s_n^1 - f|_{1,\eta_p} \to 0 \quad \& \quad |s_n^2 - g|_{1,\eta_p} \to 0, \quad \text{as} \quad n \to \infty.$$

By lemma 10.1,

$$(2) \qquad (\mathbb{E}_{\eta_p}(s_n^1), \mathbb{E}_{\eta_p}(s_n^2)) = p!(\tilde{s}_n^1, \tilde{s}_n^2).$$

By definition A.26, LHS(2) $\to (\mathbb{E}_{\eta_p}(f), \mathbb{E}_{\eta_p}(g))$, as $n \to \infty$. Next, by (1) and lemma 10.2(b),

$$|\tilde{s}_n^1 - \tilde{f}|_{2,\ell_p} \to 0 \quad \& \quad |\tilde{s}_n^2 - \tilde{g}|_{2,\ell_p} \to 0, \quad \text{as} \quad n \to \infty,$$

whence, RHS(2) $\to p!(\tilde{f}, \tilde{g})$. Thus (2) reduces to ($a$).

(b) The equality in (b) follows on taking $g = f$ in (a), and the inequality from (6.20). Thus (b).

(c) By (b) and the equality $\tilde{\tilde{f}} = \tilde{f}$,

$$|\mathbb{E}_{\eta_p}(\tilde{f}) - \mathbb{E}_{\eta_p}(f)| = \sqrt{p!}\, |(\tilde{f} - f)^{\sim}|_{2,\ell_p} = |(\tilde{\tilde{f}} - \tilde{f})|_{2,\ell_p} = 0.$$

Hence (c).

(d) We appeal to the Pettis property (A.32), (A.33) that

$$\text{LHS}(d) = \mathbb{E}_{\mathbb{P}}\left\{\int_{\mathbb{R}^p} f(t)\eta_p\,(\mathrm{d}t)\right\} = \int_{\mathbb{R}^p} f(t)\cdot(\mathbb{E}_{\mathbb{P}}\circ\eta_p)\,(\mathrm{d}t) = 0, \qquad \text{by 9.13}(a).$$

■

Note. The equality in 10.3(c) prevails only when $f \in \mathcal{P}_{1,\eta_p}$. It may happen that $\tilde{f} \in \mathcal{P}_{1,\eta_p}$ & $f \notin \mathcal{P}_{1,\eta_p}$. Take $p = 2$ and $f(t) = t_1 - t_2$ on $\mathbb{R}^2$. Then $\tilde{f} = 0$ and trivially $\tilde{f} \in \mathcal{P}_{1,\eta_2}$. But $f \notin L_2(\mathbb{R}^2)$ and hence, cf. 10.5(a) below, $f \notin \mathcal{P}_{1,\eta_2}$.

We now turn to the demarcation of the class $\mathcal{P}_{1,\eta_p}$. The next crucial lemma is obtained by careful analysis of the quasi- and semi-variations of the indefinite integral $\nu_{\eta_p,f}$ defined in (A.36). It is not, however, the best possible, and is superseded by corollary 11.4(a) below. Taking $\mathcal{D} = \mathcal{D}_p$ & $\rho = \eta_p$ in (A.12), we have $\mathcal{D}_p^{\text{loc}} = \mathcal{B}_p$. Hence the inequalities in (A.4) read

$$\forall A \in \mathcal{B}_p, \qquad q_{\nu_{\eta_p,f}}(A) \leqslant s_{\nu_{\eta_p,f}}(A) \leqslant 2q_{\nu_{\eta_p,f}}(A).$$

10.4. Main lemma.

(a) *Let* $f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$. *Then* $|f|_{1,\eta_p} \leqslant 2\sqrt{p!}\,|f|_{2,\ell_p} \leqslant \infty$.

(b) *Let* $f \in \mathcal{P}_{1,\eta_p}$. *Then* $(1/\sqrt{p!})|f|_{2,\ell_p} \leqslant |f|_{1,\eta_p} \leqslant 2\sqrt{p!}|f|_{2,\ell_p} < \infty$.

Proof. (a) Taking $\rho = \eta_p$ in the inequalities in corollary A.39, we get

$$\sup_{C\in\mathcal{D}_{\eta_p}(f)} |\mathbb{E}_{\eta_p}(f\chi_C)| \leqslant |f|_{1,\eta_p} \leqslant 2 \sup_{C\in\mathcal{D}_{\eta_p}(f)} |\mathbb{E}_{\eta_p}(f\chi_C)|.$$

Since for $C \in \mathcal{D}_{\eta_p}(f)$, $f\chi_C$ is in $\mathcal{P}_{1,\eta_p}$, therefore by 10.3(b),

$$|\mathbb{E}_{\eta_p}(f\chi_C)| = \sqrt{p!}\,|(f\chi_C)\tilde{}\,|_{2,\ell_p}.$$

Thus

$$(1) \qquad \sqrt{p!} \sup_{C\in\mathcal{D}_{\eta_p}(f)} |(f\chi_C)\tilde{}\,|_{2,\ell_p} \leqslant |f|_{1,\eta_p} \leqslant 2\sqrt{p!} \sup_{C\in\mathcal{D}_{\eta_p}(f)} |(f\chi_C)\tilde{}\,|_{2,\ell_p}.$$

But since, cf. (6.20), symmetrization is $|\cdot|_{2,\ell_p}$-norm contracting,

$$(2) \qquad |f|_{1,\eta_p} \leqslant \text{RHS}(1) \leqslant 2\sqrt{p!} \sup_{C\in\mathcal{D}_{\eta_p}(f)} |(f\chi_C)|_{2,\ell_p} \leqslant 2\sqrt{p!}\,|f|_{2,\ell_p}.$$

Thus (a).

(b) Since $f \in \mathcal{P}_{1,\eta_p}$, therefore cf. (A.37), $\mathcal{D}_{\eta_p}(f) = \mathcal{D}_p^{\text{loc}} = \mathcal{B}_p$. Hence

$$\text{LHS}(1) = \sqrt{p!} \sup_{C\in\mathcal{B}_p} |(f\chi_C)\tilde{}\,|_{2,\ell_p} \geqslant \frac{1}{\sqrt{p!}}|f|_{2,\ell_p},$$

the last very crucial step coming from the symmetrization inequality 6.22. Thus from (1) we get

$$\frac{1}{\sqrt{p!}}|f|_{2,\ell_p} \leqslant \text{LHS}(1) \leqslant |f|_{1,\eta_p}.$$

Combining this with (a), we have (b). ■

10.5. Main theorem. *Let* $p \in \mathbb{N}_+$. *Then*

(a) $\mathcal{P}_{1,\eta_p} = L_2(\mathbb{R}^p)$, *and the norms* $|\cdot|_{1,\eta_p}$, $|\cdot|_{2,\ell_p}$ *are equivalent.*

(b) $(1/\sqrt{p!})\mathbb{E}_{\eta_p}$ *is a partial isometry on* $L_2(\mathbb{R}^p)$ *into* $\mathcal{L}_2$, &

$$\textit{Null space } (\mathbb{E}_{\eta_p}) = \{f : f \in L_2(\mathbb{R}^p) \text{ \& } \tilde{f} = 0\};$$

(c) $(1/\sqrt{p!})\mathbb{E}_{\eta_p}$ *is an isometry on* $L_2^{\text{sym}}(\mathbb{R}^p)$ *into* $\mathcal{L}_2$.

Proof. (a) Let $f \in L_2(\mathbb{R}^p)$. Then by the main lemma 10.4(a), $|f|_{1,\eta_p} \leqslant 2\sqrt{p!}\,|f|_{2,\ell_p} < \infty$, whence by (A.10) and (A.17), $f \in \mathcal{P}_{1,\eta_p}$. Next, let $f \in \mathcal{P}_{1,\eta_p}$. Then, $|f|_{1,\eta_p} < \infty$, whence by the main lemma (b), $|f|_{2,\ell_p} < \infty$, i.e. $f \in L_2(\mathbb{R}^p)$. Thus (a).

(b) follows immediately from (a) by virtue of the equalities in theorem 10.3(a), (b).

(c) For $f, g \in L_2^{\text{sym}}(\mathbb{R}^p)$, theorem 10.3($a$) reduces to

$$(\mathbb{E}_{\eta_p}(f), \mathbb{E}_{\eta_p}(g)) = p!(f,g)_{2,\ell_p},$$

and yields (c). ■

From theorem 10.5(b) and rudimentary Hilbert space theory, we see that Range $\mathbb{E}_{\eta_p}$ is a closed subspace of $\mathcal{L}_2$. Since, cf. (A.29), cls Range $\mathbb{E}_{\eta_p} = \mathcal{S}_{\eta_p}$, it follows that Range $\mathbb{E}_{\eta_p} = \mathcal{S}_{\eta_p}$. Next in exact analogy with (8.4), we have

$$\forall D \in \mathcal{D}_p, \quad \exists \text{ a sequence } (P_n)_1^\infty \text{ in } \mathcal{P}_p \ni \eta_p(D) = \lim_{n\to\infty} \eta_p(P_n),$$

whence exactly as in 8.5, $\mathcal{S}_{\eta_p} = \mathfrak{S}\{\eta_p(P) : P \in P_p\}$. Thus

$$\text{(10.6)} \qquad \forall p \in \mathbb{N}_+, \qquad \text{Range}\,\mathbb{E}_{\eta_p} = \mathcal{S}_{\eta_p} = \mathfrak{S}\{\eta_p(P) : P \in P_p\}.$$

Combining (10.6) and main theorem 10.5, we see that:

$$\text{(10.7)} \qquad \begin{cases} (1/\sqrt{p!})\mathbb{E}_{\eta_p} \text{ is a partial isometry on } L_2(\mathbb{R}^p) \text{ onto } \mathcal{S}_{\eta_p}, \\ (1/\sqrt{p!})\mathbb{E}_{\eta_p} \text{ is an isometry on } L_2^{\text{sym}}(\mathbb{R}^p) \text{ onto } \mathcal{S}_{\eta_p}. \end{cases}$$

A considerable improvement of the inequalities in 10.4(b) results when $f \geqslant 0$. The absolute value $|\mathbb{E}_{\eta_p}(f)|$ comes into the picture, as the next result shows:

10.8. Theorem. *Let $f \in \mathcal{P}_{1,\eta_p}$ and $f(\cdot) \geqslant 0$. Then $|f|_{2,\ell_p} \leqslant |\mathbb{E}_{\eta_p}(f)| \leqslant |f|_{1,\eta_p}$.*

Proof. The second equality is covered by (A.27). The proof of the first hinges on the result 9.13(b), to wit,

$$\text{(1)} \qquad \forall D \in \mathcal{D}_p, \qquad \ell_p(D) \leqslant |\eta_p(D)|^2.$$

Case 1. Let $f = \sum_{k=1}^r a_k \chi_{D_k} \in S(\mathcal{D}_p, \mathbb{R}_{0+})$. Then

$$\begin{aligned} |\mathbb{E}_{\eta_p}(f)|^2 &= \left|\sum_{k=1}^r a_k \eta_p(D_k)\right|^2 \\ \text{(2)} \qquad &= \sum_{k=1}^r a_k^2 |\eta_p(D_k)|^2 + \sum_{i=1}^{r-1} \sum_{j=i+1}^r (a_i \eta_p(D_k), a_j \eta_p(D_j)). \end{aligned}$$

Now, by 9.13(a),

$$\begin{aligned} (a_i\eta_p(D_i), a_j\eta_p(D_j)) &= a_i a_j (\eta_p(D_i), \eta_p(D_j)) \\ &= a_i a_j \sum_{\phi \in \text{Perm}(p)} \ell_p(D_i \cap D_j^\phi) \geqslant 0, \qquad \text{since } a_i, a_j \geqslant 0. \end{aligned}$$

Hence by (2) and (1),

$$|\mathbb{E}_{\eta_p}(f)|^2 \geqslant \sum_{k=1}^r a_k^2 |\eta_p(D_k)|^2 + 0 \geqslant \sum_{k=1}^r a_k^2 \ell_p(D_k) = |f|_{2,\ell_p}^2.$$

Thus the result holds in Case 1.

Case 2. Let $f \in \mathcal{P}_{1,\eta_p}$ and $f \geqslant 0$. Then a straightforward simple function approximation based on theorem A.24(*b*) yields

$$|f|^2_{2,\ell_p} \leqslant |\mathbb{E}_{\eta_p}(f)|^2,$$

i.e. we again get the first equality. ■

11. The projection theorem and the formula for the orthogonal decomposition of $\mathcal{L}_2^{(\xi)}$

From the orthogonal decomposition (9.20) we see that

$$\forall x \in \mathcal{L}_2, \qquad \operatorname{proj}(x|\mathcal{L}_2^{(\xi)}) = \sum_{p=0}^{\infty} \operatorname{proj}(x|\mathcal{S}_{\eta_p}).$$

But, cf. (10.6), $\operatorname{proj}(x|\mathcal{S}_{\eta_p}) \in \operatorname{Range}\mathbb{E}_{\eta_p}$. Thus $\operatorname{proj}(x|\mathcal{S}_{\eta_p}) = \mathbb{E}_{\eta_p}(f_x^p)$, where, cf. 10.5(*a*), $f_x^p \in L_2(\mathbb{R}^p)$. In the next theorem we exhibit the function f_x^p explicitly in terms of x and η_p. As with CAOS measures, cf. Masani (1968, 5.10), this theorem is best stated as a projection theorem involving Radon–Nikodym derivatives:

11.1. Projection theorem. *Let* $x \in \mathcal{L}_2$, $p \in \mathbb{N}_+$ & $\forall \Delta \in \mathcal{D}_p$, $\nu_x(\Delta) := (x, \eta_p(\Delta))$. *Then*

(*a*) $\nu_x \in \operatorname{CA}(\mathcal{D}_p, \mathbb{R})$ & $\nu_x \prec\!\!\prec \ell_p$;

(*b*) $\mathrm{d}\nu_x/\mathrm{d}\ell_p \in L_2^{\mathrm{sym}}(\mathbb{R}^p)$;

(*c*)

$$\operatorname{proj}(x|\mathcal{S}_{\eta_p}) = \frac{1}{p!}\int_{\mathbb{R}^p} \frac{\mathrm{d}\nu_x}{\mathrm{d}\ell_p}(t)\cdot \eta_p\,(\mathrm{d}t);$$

(*d*)

$$|\operatorname{proj}(x|\mathcal{S}_{\eta_p})|^2 = \frac{1}{p!}\int_{\mathbb{R}^p} \left|\frac{\mathrm{d}\nu_x}{\mathrm{d}\ell_p}(t)\right|^2 \ell_p\,(\mathrm{d}t).$$

Proof. (*a*) Since $\eta_p \in \operatorname{CA}(\mathcal{D}_p, \mathcal{L}_2)$ and $\eta_p \prec\!\!\prec \ell_p$, therefore (*a*) is obvious.

(*b*) Let $\hat{x} := \operatorname{proj}(x|\mathcal{S}_{\eta_p})$. Then by (10.6), $\hat{x} \in \operatorname{Range}\mathbb{E}_{\eta_p}$. Hence, cf. theorem 10.5(*a*),

(1) $$\exists f \in \mathcal{P}_{1,\eta_p} = L_2(\mathbb{R}^p) \quad \ni \quad \hat{x} = \mathbb{E}_{\eta_p}(f).$$

Now, by theorem 10.3(*a*) and 6.19, $\forall \Delta \in \mathcal{D}_p$,

$$(x, \eta_p(\Delta)) = (\hat{x}, \eta_p(\Delta)) = (\mathbb{E}_{\eta_p}(f), \mathbb{E}_{\eta_p}(\chi_\Delta)) = p!(\tilde{f}, \tilde{\chi}_\Delta)_{\ell_p} = p!(\tilde{f}, \chi_\Delta)_{\ell_p}.$$

Thus

$$\forall \Delta \in \mathcal{D}_p, \qquad \nu_x(\Delta) := (x, \eta_p(\Delta)) = p!\int_\Delta \tilde{f}(t)\ell_p\,(\mathrm{d}t).$$

It follows from the Radon–Nikodym theorem that for ℓ_p almost all $t \in \mathbb{R}^p$,

(3) $$\frac{\mathrm{d}\nu_x}{\mathrm{d}\ell_p}(t) = p!\tilde{f}(t).$$

By (1) and (3), $\mathrm{d}\nu_x/\mathrm{d}\ell_p \in L_2^{\mathrm{sym}}(\mathbb{R}^p)$. Thus (*b*).

(*c*) Combining (1), theorem 10.3(*c*) and (3), we see that $\exists f \in L_2(\mathbb{R}^p)$ such that

$$\hat{x} = \mathbb{E}_{\eta_p}(f) = \mathbb{E}_{\eta_p}(\tilde{f}) = \frac{1}{p!}\mathbb{E}_{\eta_p}\left(\frac{\mathrm{d}\nu_x}{\mathrm{d}\ell_p}\right).$$

This gives (c). (d) follows from (c) and theorem 10.3(b) for symmetric f. ■

Since from (9.20) we have

$$\forall x \in \mathcal{L}_2, \qquad \operatorname{proj}\{x|\mathcal{L}_2^{(\xi)}\} = \sum_{k=0}^{\infty} \operatorname{proj}(x|\mathcal{S}_{\eta_p}),$$

we at once get as a corollary of the projection theorem the following explicit version of the canonical resolution of $\mathcal{L}_2^{(\xi)}$ in terms of homogeneous chaoses:

11.2. Corollary. (The orthogonal decomposition of $\mathcal{L}_2$) *Let* $\forall x \in \mathcal{L}_2$ & $\forall p \in \mathbb{N}_+$,

$$\nu_x^p(\cdot) := (x, \eta_p(\cdot)) \quad on \quad \mathcal{D}_p \quad \& \quad f_x^p := \frac{\mathrm{d}\nu_x^p}{\mathrm{d}\ell_p}, \qquad a.e.\ \ell_p\ on\ \mathbb{R}^p.$$

Then

(a) $\forall x \in \mathcal{L}_2$,

$$\operatorname{proj}\{x|\mathcal{L}_2^{(\xi)}\} = \sum_{p=0}^{\infty} \frac{1}{p!} \mathbb{E}_{\eta_p}(f_x^p),$$

the decomposition being orthogonal;

(b) *in case* $\mathcal{L}_2 = \mathcal{L}_2^{(\xi)}$, *we have* $\forall x \in \mathcal{L}_2$,

$$x = \sum_{p=0}^{\infty} \frac{1}{p!} \mathbb{E}_{\eta_p}(f_x^p).$$

11.3. *Remarks.* (On Wiener's 1958 approach) A comparison of the last expansion with that of Wiener, namely,

(1) $$F(\cdot) = \lim_{n\to\infty} \sum_{p=0}^{n} G_p(K_p, \cdot), \qquad (1958,\ (3.44)\ \text{or}\ (4.7)),$$

where F in $\mathcal{L}_2$ is arbitrary, and the K_p are functions in $L_2^{\text{sym}}(\mathbb{R}^p)$, uniquely determined by F, shows that

$$G_p(K_p, \cdot) = \frac{1}{p!}[\mathbb{E}_{\eta_p}(K_p)](\cdot).$$

Indeed, to get the kernels K_p corresponding to a given F in $\mathcal{L}_2$, Wiener takes the RN derivatives with respect to ℓ_p of the scalar measures obtained taking the inner product of F with the $\mathcal{L}_2$-valued measures $\rho_p(P)(\cdot) := G_p(\chi_P, \cdot)$, where $P \in \mathcal{P}_p$, in accord with our corollary 11.2 (cf. Wiener 1958, pp. 41, 42, eqs (4.9)–(4.13)). However, to get his G_p's, i.e. our $\mathbb{E}_{\eta_p}$'s, Wiener adopted the most natural approach of starting from the 'multiple Wiener integrals', cf. § 1 (which we are able to introduce only in § 14 below)), and 'ortho-normalizing' them by taking their linear combinations in the spirit of the Gram–Schmidt process (cf. Wiener 1958, pp. 28–36). But these combinations get very unwieldy as p increases, and Wiener after getting G_1, G_2, G_3, had to assume without proof that the G_p exists for $p > 3$ (cf. Wiener 1958, p. 36, last para), as we noted in § 1. We will show their existence in § 16.

The projection theorem also allows us to dispense with the restriction of '$f \in \mathcal{P}_{1,\eta_p}$' and with the spurious factor 2, which marred the inequalities between $|f|_{1,\eta_p}$ and $|f|_{2,\ell_p}$ given in lemma 10.4(b). We have

11.4. Corollary. *Let* $f \in \mathcal{M}(\mathcal{D}_p^{\rm loc}, B\ell(\mathbb{R}))$. *Then*
(*a*) $(1/\sqrt{p!})|f|_{2,\ell_p} \leqslant |f|_{1,\eta_p} \leqslant \sqrt{p!}\,|f|_{2,\ell_p} \leqslant \infty$;
(*b*) *when* f *is symmetric,* $|f|_{1,\eta_p} = \sqrt{p!}\,|f|_{2,\ell_p} \leqslant \infty$;
(*c*) *in general,* $|\tilde{f}|_{1,\eta_p} = \sqrt{p!}\,|\tilde{f}|_{2,\ell_p}$.

Proof. (*a*) Let $x' \in (\mathcal{L}_2)'$. Then there exists $x \in \mathcal{L}_2$ such that $x'(\cdot) = (\cdot, x)_{\mathcal{L}_2} := (x, \cdot)_{\mathcal{L}_2}$. It follows that

$$\forall D \in \mathcal{D}_p, \qquad x' \circ \eta_p(D) = (\eta_p(D), x) =: \nu_x(D),$$

and hence by [MN, I, 2.32 & 2.34],

$$(1) \qquad \int_{\mathbb{R}^p} |f(t)| \cdot |x' \circ \eta_p|\,(\mathrm{d}t) = \int_{\mathbb{R}^p} |f(t)| \cdot \left|\frac{\mathrm{d}\nu_x}{\mathrm{d}\ell_p}(t)\right| \ell_p\,(\mathrm{d}t) \leqslant |f|_{2,\ell_p} \cdot \left|\frac{\mathrm{d}\nu_x}{\mathrm{d}\ell_p}\right|_{2,\ell_p}.$$

But by theorem 11.1(*d*),

$$(2) \qquad \left|\frac{\mathrm{d}\nu_x}{\mathrm{d}\ell_p}\right|_{2,\ell_p} = \sqrt{p!}\,|\,\mathrm{proj}(x|\mathcal{S}_{\eta_p})| \leqslant \sqrt{p!}\,|x|.$$

From (1) and (2) we see that

$$|f|_{1,\eta_p} := \sup_{|x'|\leqslant 1} \int_{\mathbb{R}^p} |f(t)| \cdot |x' \circ \eta_p|\,(\mathrm{d}t)$$

$$(3) \qquad \leqslant |f|_{2,\ell_p} \cdot \sup_{|x'|\leqslant 1} \left|\frac{\mathrm{d}\nu_x}{\mathrm{d}\ell_p}\right|_{2,\ell_p} \leqslant |f|_{2,\ell_p} \cdot \sqrt{p!} \cdot 1.$$

Now if $f \in \mathcal{P}_{1,\eta_p}$, then combining (3) with the first inequality in 10.4(*b*), we get the desired inequalities. But if $f \notin \mathcal{P}_{1,\eta_p}$, then these inequalities hold trivially since by theorem 10.4, $f \notin L_2(\mathbb{R}^p)$, and $|f|_{2,\ell_p} = \infty = |f|_{1,\eta_p}$. Thus (*a*).

(*b*) When $f \in \mathcal{P}_{1,\eta_p}$ is symmetric we have from 10.3(*b*) and part (*a*),

$$\sqrt{p!}\,|f|_{2,\ell_p} = \sqrt{p!}\,|\tilde{f}|_{2,\ell_p} = |\mathbb{E}_{\eta_p}(f)| \leqslant |f|_{1,\eta_p} \leqslant \sqrt{p!}\,|f|_{2,\ell_p},$$

i.e. (*b*) holds. But it also holds trivially for $f \notin \mathcal{P}_{1,\eta_p}$. Thus (*b*). The result (*c*) of course follows. ■

11.5. *Remarks.* That there can be no obvious equality connecting $|f|_{1,\eta_p}$ and $|f|_{2,\ell_p}$ is evident from the fact that whereas for symmetric $f \in \mathcal{P}_{1,\eta_p}$, we have by 11.4(*b*),

$$|f|_{1,\eta_p} = \sqrt{p!}\,|f|_{2,\ell_p},$$

we can have $|f|_{1,\eta_p} = |f|_{2,\ell_p}$ for suitable non-symmetric f. To see this, take $p = 2$, and $f = \chi_D$, where $D \in \mathcal{D}_2$ and $D \| D^\phi$, where $\mathrm{Perm}(2) = \{I, \phi\}$. Then it readily follows from (A.9), (A.3), 9.13(*b*) and 9.13(*a*), that

$$|\chi_D|_{1,\eta_2} = s_{\eta_2}(D) = |\eta_2(D)| = \sqrt{\ell_2(D)} = |\chi_D|_{2,\ell_2}.$$

The projection theorem has other important consequences, to explore which we need the following useful result:

11.6. Lemma. *Let* $p, q \in \mathbb{N}_+$ *be such that* $q \leqslant p$ *and* $p - q$ *is even. Then*

$$\forall D \in \mathcal{D}_p, \qquad \gamma^p_{(p-q)/2}(D, \cdot) \in \mathcal{P}_{1,\eta_q}$$

&

$$\forall \Delta \in \mathcal{D}_q, \qquad (\xi_p(D), \eta_q(\Delta)) = \sum_{\phi \in \mathrm{Perm}(q)} \int_{\mathbb{R}^q} \gamma^p_{(p-q)/2}(D, h)\chi_\Delta(h^\phi)\ell_q\,(\mathrm{d}h).$$

Proof. Let $D \in \mathcal{D}_p$. Then by 4.16(b) and 10.5(a), $\gamma^p_{(p-q)/2}(D, \cdot) \in L_2(\mathbb{R}^q) = \mathcal{P}_{1,\eta_q}$. Next write $x := \xi_p(D)$. Then, cf. (9.11),

$$(1) \quad \forall \Delta \in \mathcal{D}_q, \qquad \nu_x(\Delta) := (x, \eta_q(\Delta)) = (\xi_p(D), \eta_q(\Delta)) = (\xi_p(D), \xi_q(\Delta \backslash I^q_1)).$$

Fix $\Delta \in \mathcal{D}_q$. Then by (1) and the covariance equality, cf. 5.3 and 5.7,

$$(2) \qquad \nu_x(\Delta) = \sum_{k=0}^{[q/2]} \Gamma^{pq}_k(D, \Delta \backslash I^q_1).$$

Now by 5.1,

$$\Gamma^{pq}_0(D, \Delta\backslash I^q_1) = \sum_{\phi \in \mathrm{Perm}(q)} \int_{\mathbb{R}^q} \gamma^p_{(p-q)/2}(D, h)\chi_{\Delta\backslash I^q_1}(h^\phi)\ell_q\,(\mathrm{d}h)$$

$$(3) \qquad = \sum_{\phi \in \mathrm{Perm}(q)} \int_{\mathbb{R}^q} \gamma^p_{(p-q)/2}(D, h)\chi_\Delta(h^\phi)\ell_q\,(\mathrm{d}h),$$

by the ℓ_q-negligibility of I^q_1. Now let $k \in [1, [q/2]]$. Since by lemma 5.2(b), $\Gamma^{pq}_k(D, \cdot)$ has I^q_k as a carrier and $I^q_k \subseteq I^q_1$, it follows that $\Gamma^{pq}_k(D, \Delta\backslash I^q_1) = 0$. Thus the summation in (2) contains only the zeroth term, which is given in (3). Hence the result. ∎

This lemma in conjunction with the projection theorem 11.1 gives us the following theorem:

11.7. Theorem. *Let $p \in \mathbb{N}_+$ & $k \in [1, [p/2]]$. Then, cf. (9.15),*

$$\forall D \in \mathcal{D}_p, \qquad \xi_{p,k}(D) := \mathrm{proj}(\xi_p(D)|\mathcal{S}_{\eta_{p-2k}}) = \int_{\mathbb{R}^{p-2k}} \gamma^p_k(D, h)\eta_{p-2k}\,(\mathrm{d}h).$$

Proof. Let $D \in \mathcal{D}_p$ and write $x := \xi_p(D)$ and $\hat{x} := \mathrm{proj}(x|\mathcal{S}_{\eta_{p-2k}})$. Then, taking $q = p-2k$, we change variables in the integral in 11.6, by letting $t = h^\phi$ and $h = t^{\phi^{-1}}$. Thus $\forall \Delta \in \mathcal{D}_{p-2k}$,

$$\int_{\mathbb{R}^{p-2k}} \gamma^p_k(D, h)\chi_\Delta(h^\phi)\ell_{p-2k}\,(\mathrm{d}h) = \int_{\mathbb{R}^{p-2k}} \gamma^p_k(D, t^{\phi^{-1}})\chi_\Delta(t)\ell_{p-2k}\,(\mathrm{d}t),$$

since ℓ_{p-2k} is a permutation invariant. It follows from the last lemma that $\forall \Delta \in \mathcal{D}_{p-2k}$,

$$\nu_x(\Delta) := (x, \eta_{p-2k}(\Delta)) = \sum_{\phi \in \mathrm{Perm}(p-2k)} \int_\Delta \gamma^p_k(D, t^{\phi^{-1}})\ell_{p-2k}\,(\mathrm{d}t).$$

Hence by the Radon–Nikodym theorem,

$$(1) \qquad \frac{\mathrm{d}\nu_x}{\mathrm{d}\ell_{p-2k}}(t) = \sum_{\phi \in \mathrm{Perm}(p-2k)} \gamma^p_k(D, t^{\phi^{-1}}) = (p-2k)!\tilde{\gamma}^p_k(D, t), \quad \text{a.e.} \quad (\ell_p),$$

where $\tilde{\gamma}_k^p(D,\cdot)$ is the symmetrization of $\gamma_k^p(D,\cdot)$. Applying successively the projection theorem 11.1(c) and (1), and 10.3(c), we get

$$\hat{x} = \frac{1}{(p-2k)!}\int_{\mathbb{R}^{p-2k}} \frac{\mathrm{d}\nu_x}{\mathrm{d}\ell_{p-2k}}(t)\eta_{p-2k}(\mathrm{d}t) = \int_{\mathbb{R}^{p-2k}} \tilde{\gamma}_k^p(D,t)\eta_{p-2k}(\mathrm{d}t)$$
$$= \int_{\mathbb{R}^{p-2k}} \gamma_k^p(D,t)\eta_{p-2k}(t).$$

■

11.8. *Remarks.* The equation (1) in the last proof shows that the symmetrization of the canonical coefficient $\gamma_k^p(D,\cdot)$, introduced in (4.17), is equal to $1/(p-2k)!$ times the RN derivative of the measure $\nu(\cdot) := (\xi_p(D), \eta_{p-2k}(\cdot))$ with respect to ℓ_{p-2k}.

From theorem 11.7 and the orthogonal decomposition of $\mathcal{S}_{\xi_p}$ given in (9.9), we readily infer the following formulae for the projection of $\xi_p(D)$ on $\mathcal{S}_{\xi_q}$:

11.9. Corollary. *Let $p, q \in \mathbb{N}_+$ be such that $q \leqslant p$ and $p-q$ is even. Then, cf. 9.1,*

$$\forall D \in \mathcal{D}_p, \quad \zeta_{pq}(D) := \operatorname{proj}(\xi_p(D)|\mathcal{S}_{\xi_q}) = \sum_{k=0}^{[q/2]} \int_{\mathbb{R}^{q-2k}} \gamma^p_{\frac{1}{2}(p-q)+k}(D,h)\eta_{q-2k}(\mathrm{d}h).$$

Proof. From (9.9) it follows, with an obvious notation, that

(1) $$P_{\mathcal{S}_{\xi_q}} = P_{\mathcal{S}_{\eta_q}} + P_{\mathcal{S}_{\eta_{q-2}}} + \cdots,$$

the final term being $P_{\mathcal{S}_{\eta_1}}$ or $P_{\mathcal{S}_{\eta_0}}$ according as q is odd or even. Hence,

$$\operatorname{proj}(\xi_p(D)|\mathcal{S}_{\xi_q}) = \sum_{k=0}^{[q/2]} \operatorname{proj}(\xi_p(D)|\mathcal{S}_{\eta_{q-2k}})$$
$$= \sum_{k=0}^{[q/2]} \int_{\mathbb{R}^{q-2k}} \gamma^p_{(p-(q-2k))/2}(D,h)\eta_{q-2k}(\mathrm{d}h) \qquad \text{by 11.7}$$
$$= \sum_{k=0}^{[q/2]} \int_{\mathbb{R}^{q-2k}} \gamma^p_{\frac{1}{2}(p-q)+k}(D,h)\eta_{q-2k}(\mathrm{d}h).$$

■

11.10. Corollary. *Let $p \in \mathbb{N}_+$. Then $\forall D \in \mathcal{D}_p$,*

(a)

$$\xi_p(D) = \sum_{k=0}^{[p/2]} \int_{\mathbb{R}^{p-2k}} \gamma_k^p(D,h)\eta_{p-2k}(\mathrm{d}h);$$

(b)

$$\zeta_p(D) := \operatorname{proj}(\xi_p(D)|\mathcal{S}_{\xi_{p-2}}) = \sum_{k=1}^{[p/2]} \int_{\mathbb{R}^{p-2k}} \gamma_k^p(D,h)\eta_{p-2k}(\mathrm{d}h).$$

Proof. (a) Put $q = p$ in 11.9 and note that $\zeta_{pp}(D) = \xi_p(D)$. (b) follows from (a) on noting that the $k = 0$ term on the RHS is just η_p since $\lambda_0^p(D,h) = \chi_D(h)$. ■

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The projection equalities enable us to answer the question raised at the tail end of § 5, as to the exact relationship between η_{p+q} and the product measure $\eta_p \times \eta_q$. In this problem a significant part is played by the fragment:

$$J_1^{p+q} := (I_1^p \times \mathbb{R}^q) \cup (\mathbb{R}^p \times I_1^q), \tag{11.11}$$

of the diagonal skeleton I_1^{p+q}, cf. (4.2), and by the obvious set-theoretical triviality concerning it, to wit:

$$\forall p, q \in \mathbb{N}_+,\ \forall E \subseteq \mathbb{R}^p\ \&\ \forall F \subseteq \mathbb{R}^q, \quad (E \times F)\backslash J_1^{p+q} = (E\backslash I_1^p) \times (F\backslash I_1^q). \tag{11.12}$$

11.13. Proposition. *Let $p, q \in \mathbb{N}_+$, $D \in \mathcal{D}_{p+q}$ and J_1^{p+q} be as in (11.11). Then*

$$(\eta_p \times \eta_q)(D) - \eta_{p+q}(D) = \zeta_{p+q}(D\backslash J_1^{p+q}).$$

Proof. First, let $D := E \times F \in \mathcal{D}_p \times \mathcal{D}_q$. Then using the results (9.11) and writing $J := J_1^{p+q}$,

$$\begin{aligned}(\eta_p \times \eta_q)(D) &:= \eta_p(E)\eta_q(F) = \xi_p(E\backslash I_1^p) \cdot \xi_q(F\backslash I_1^q) = \xi_{p+q}\{(E\backslash I_1^p) \times (F\backslash I_1^q)\} \\ &= \xi_{p+q}\{(E \times F)\backslash J\} = \eta_{p+q}(D\backslash J) + \zeta_{p+q}(D\backslash J), \qquad \text{by (11.12)}.\end{aligned} \tag{1}$$

Now since $J \subseteq I_1^{p+q} \in \mathcal{N}_{\ell_{p+q}}$ and $\eta_{p+q} \ll \ell_{p+q}$, therefore $\eta_{p+q}(D \cap J) = 0$. Hence (1) reduces to the desired equality:

$$(\eta_p \times \eta_q)(D) = \eta_{p+q}(D) + \zeta_{p+q}(D\backslash J). \tag{2}$$

Next let $\forall D \in \mathcal{D}_{p+q}$, $\rho(D) := \zeta_{p+q}(D\backslash J)$. Then the result (2) can be restated as

$$\eta_p \times \eta_q - \eta_{p+q} = \rho \quad \text{on} \quad \mathcal{D}_p \times \mathcal{D}_q. \tag{3}$$

Since $\eta_p = \xi_p^a$ and $\zeta_p = \xi_p^b$, it follows readily from 5.26 that

$$\eta_p \times \eta_q - \eta_{p+q} \quad \& \quad \rho \text{ are in } \mathrm{CA}(\mathcal{D}_{p+q}, \mathcal{L}_2). \tag{4}$$

By (3), (4) and the identity principle A.8, the equality (3) holds on δ-ring$(\mathcal{D}_p \times \mathcal{D}_q)$, i.e. on $\mathcal{D}_{p+q}$. ■

The decomposition of $\eta_p \times \eta_q$ yielded by the last proposition is its Lebesgue decomposition with respect to ℓ_{p+q}. More precisely,

11.14. Corollary. (Lebesgue decomposition of $\eta_p \times \eta_q$) *Let $p, q \in \mathbb{N}_+$ & $(\eta_p \times \eta_q)^a$ and $(\eta_p \times \eta_q)^b$ be the absolutely continuous and singular parts of $\eta_p \times \eta_q$ with respect to ℓ_{p+q}, respectively. Then*

$$(\eta_p \times \eta_q)^a(\cdot) = \eta_{p+q}(\cdot), \qquad (\eta_p \times \eta_q)^b(\cdot) = \zeta_{p+q}(\cdot\backslash J_1^{p+q}).$$

Proof. Let $\rho(\cdot) := \zeta_{p+q}(\cdot\backslash J_1^{p+q})$ on $\mathcal{D}_{p+q}$. Then by 11.13,

$$\eta_p \times \eta_q = \eta_{p+q} + \rho. \tag{1}$$

Grant momentarily that

$$I_1^{p+q} \text{ is a carrier of } \rho. \tag{I}$$

Then since by 7.1(d), $I_1^{p+q} \in \mathcal{N}_{\ell_{p+q}}$, it follows that $\rho(\cdot)$ is singular respect to ℓ_{p+q}. Also by 9.12, $\eta_{p+q} \ll \ell_{p+q}$. Hence from the uniqueness of the Lebesgue decomposition, we infer that $(\eta_p \times \eta_q)^a = \eta_{p+q}$ and $(\eta_p \times \eta_q)^b = \rho$. It remains to prove (I).

Proof of (I). Let $D \in \mathcal{D}_{p+q}$. Then

$$\begin{aligned}\rho(D \cap I_1^{p+q}) &:= \zeta_{p+q}[(D \cap I_1^{p+q}) \backslash J_1^{p+q}] = \zeta_{p+q}[D \cap (I_1^{p+q} \backslash J_1^{p+q})] \\ &= \xi_{p+q}[D \cap (I_1^{p+q} \backslash J_1^{p+q}) \cap I_1^{p+q}] \qquad \text{by (9.11)} \\ &= \xi_{p+q}[D \cap (I_1^{p+q} \backslash J_1^{p+q})] = \xi_{p+q}[(D \backslash J_1^{p+q}) \cap I_1^{p+q}] \\ &=: \zeta_{p+q}[D \backslash J_1^{p+q}] =: \rho(D) \qquad \text{by (9.11).}\end{aligned}$$

Thus by the definition A.2 of carrier, we have (I). ■

We can of course expand $\zeta_{p+q}(D \backslash J_1^{p+q})$, the RHS in 11.13, by the projection theorem 11.10(b); thus writing $J = J_1^{p+q}$, we get

$$(*) \qquad \zeta_{p+q}(D \backslash J) = \sum_{r=1}^{[(p+q)/2]} \int_{\mathbb{R}^{p+q-2r}} \gamma_{p+q-2r}^{p+q}(D \backslash J, h) \eta_{p+q-2r}(\mathrm{d}h).$$

But the exclusion of J on the RHS results in considerable simplification. Because of vanishing, the upper terminus of the $\sum$ drops to $\min\{p, q\}$, and the J can be deleted from the integrand on the RHS. Moreover, in the evaluation of the canonical coefficients $\gamma_k^{p+q}(D \backslash J, h)$, we can ignore all partitions having a cell in either $[1, p \vee q]$ or in $[(p \vee q)+1, p+q]$. To show all this conveniently, we introduce the following notation for the subclass of partitions in Π_r^{p+q}, for which all cells $\{i, j\}$ satisfy $i \leqslant p \vee q \leqslant j$:

11.15. *Notation.* Let $p, q \in \mathbb{N}_+$ & $r \in [1, [(p+q)/2]]$. Then

$$\mathring{\Pi}_r^{p+q} := \{\pi : \pi \in \Pi_r^{p+q} \ \& \ \forall \Delta \in \pi, \min \Delta \leqslant p \vee q < \max \Delta\};$$

Crucial to the simplification of $(*)$ are the following properties of $\mathring{\Pi}_r^{p+q}$, the proofs of which are straightforward, and left to the reader.

11.16. Triviality. *Let p, q, r be as in 11.15. Then*

(a) $\mathring{\Pi}_r^{p+q} \neq \emptyset$ *iff* $r \in [1, p \wedge q]$;

(b) $\forall \pi \in \Pi_r^{p+q} \backslash \mathring{\Pi}_r^{p+q}$, $I(\pi, p+q) \subseteq J_1^{p+q}$.

We now assert the following lemma:

11.17. Main lemma. *Let $p, q \in \mathbb{R}_+$ & $q \leqslant p$. Then $\forall D \in \mathcal{D}_{p+q}$,*

$$\zeta_{p+q}(D \backslash J_1^{p+q}) = \sum_{r=1}^{q} \sum_{\pi \in \mathring{\Pi}_r^{p+q}} \int_{\mathbb{R}^{p+q-2r}} \lambda_\pi^{p+q}(D, h) \eta_{p+q-2r}(\mathrm{d}h).$$

Proof. Let $D \in \mathcal{D}_{p+q}$. Then by 11.10(b), writing $J = J_1^{p+q}$, we have

$$(1) \qquad \zeta_{p+q}(D \backslash J) = \sum_{r=1}^{[(p+q)/2]} \int_{\mathbb{R}^{p+q-2r}} \gamma_r^{p+q}(D \backslash J, h) \eta_{p+q-2r}(\mathrm{d}h).$$

Now grant momentarily that

(I) $\quad \forall r \in [1, [(p+q)/2]] \quad \& \quad \forall \pi \in \Pi_r^{p+q} \backslash \mathring{\Pi}_r^{p+q}$,

$$\lambda_\pi^{p+q}(D \backslash J, \cdot) = 0 \quad \text{on} \quad \mathbb{R}^{p+q-2r}.$$

(II) $\forall r \in [1,q]$ & $\forall \pi \in \mathring{\Pi}_r^{p+q}$,

$$\lambda_\pi^{p+q}(D \cap J, \cdot) = 0 \quad \text{on} \quad \mathbb{R}_*^{p+q-2r} := \mathbb{R}^{p+q-2r} \backslash I_1^{p+q-2r} \qquad \text{cf. (4.1).}$$

Then $\forall r \in [1, [(p+q)/2]]$ & $\forall h \in \mathbb{R}^{p+q-2r}$,

$$(2) \quad \gamma_r^{p+q}(D\backslash J, h) := \sum_{\pi \in \Pi_r^{p+q}} \lambda_\pi^{p+q}(D\backslash J, h) = \sum_{\pi \in \mathring{\Pi}_r^{p+q}} \lambda_\pi^{p+q}(D\backslash J, h), \qquad \text{by (I).}$$

But, cf. 11.16(a), $\mathring{\Pi}_r^{p+q} \neq \emptyset$, only for $r \leqslant p \wedge q = q$. Hence by (2),

$$\sum_{r=1}^{[(p+q)/2]} \gamma_r^{p+q}(D\backslash J, h) = \sum_{r=1}^{[(p+q)/2]} \sum_{\pi \in \mathring{\Pi}_r^{p+q}} \gamma_\pi^{p+q}(D\backslash J, h) = \sum_{r=1}^{q} \sum_{\pi \in \mathring{\Pi}_r^{p+q}} \lambda_\pi^{p+q}(D\backslash J, h)$$

$$= \sum_{r=1}^{q} \sum_{\pi \in \mathring{\Pi}_r^{p+q}} \lambda_\pi^{p+q}(D, h), \quad h \notin I_1^{p+q-2r}, \quad \text{by (II).}$$

Since $I_1^{p+q-2r} \in \mathcal{N}_{\ell_{p+q-2r}} = \mathcal{N}_{\eta_{p+q-2r}}$, the last equality holds for η_{p+q-2r} almost all h. Integrating, we get

$$\sum_{r=1}^{[(p+q)/2]} \int_{\mathbb{R}^{p+q-2r}} \gamma_r^{p+q}(D\backslash J, h)\eta_{p+q-2r}\,(\mathrm{d}h) = \sum_{r=1}^{q} \sum_{\pi \in \mathring{\Pi}_r^{p+q}} \int_{\mathbb{R}^{p+q-2r}} \lambda_\pi^{p+q}(D, h)\eta_{p+q-2r}\,(\mathrm{d}h).$$

Substituting in (1) we get the desired expression for $\zeta_{p+q}(D\backslash J)$.

It only remains to justify (I) and (II).

Proof of (I). Let $r \in [1, [(p+q)/2]]$, $\pi \in \Pi_r^{p+q} \backslash \mathring{\Pi}_r^{p+q}$ and $h \in \mathbb{R}^{p+q-2r}$. Then by (4.7) and 11.16(b),

$$I_\pi(h) \subseteq I(\pi, p+q) \subseteq J.$$

Thus $(D\backslash J) \cap I_\pi(h) = \emptyset$, and therefore certainly

$$\lambda_\pi^{p+q}(D\backslash J, h) := \ell_k[\wp_{\pi^*}(\emptyset)] = 0.$$

Thus (I).

Proof of (II). Let $r \in [1,q]$ & $\pi \in \mathring{\Pi}_r^{p+q}$. Then ${}^*\pi \subsetneqq [1,p]$ & $\pi^* \subsetneqq [p+1, p+q]$. But J is the union of all $I_{\bar{\Delta}}^{p+q}$, where $\bar{\Delta} \subseteq [1,p]$ or $\bar{\Delta} \subseteq [p+1, p+q]$, and where consequently $\bar{\Delta} \notin \pi$. Hence by lemma 7.3(b),

$$\forall h \in \mathbb{R}_*^{p+q-2r}, \qquad \lambda_\pi^{p+q}(D \cap I_{\bar{\Delta}}^{p+q}, h) = 0.$$

Since J is a finite union of such $I_{\bar{\Delta}}^{p+q}$, and $\lambda(\cdot, h)$ is FA, it follows readily that

$$\lambda_\pi^{p+q}(D \cap J, h) = 0, \qquad \forall h \in \mathbb{R}_*^{p+q-2r}.$$

Thus (II). ∎

Combining lemmas 11.13 and 11.17, we get the following theorem, which gives the exact nexus between $\eta_p \times \eta_q$ and η_{p+q}, and which will play an important role in §§ 14 and 17.

11.18. Theorem. *Let $p, q \in \mathbb{N}_+$ & $q \leqslant p$. Then $\forall D \in \mathcal{D}_{p+q}$,*

$$(\eta_p \times \eta_q)(D) - \eta_{p+q}(D) = \sum_{r=1}^{q} \sum_{\pi \in \overset{\circ}{\Pi}{}_r^{p+q}} \int_{\mathbb{R}^{p+q-2r}} \lambda_\pi^{p+q}(D,h)\eta_{p+q-2r}\,(\mathrm{d}h).$$

12. The π, h sectioning of functions

Before we can turn to integration with respect to the Wiener vector measure ξ_p, it is necessary to consider the sectioning of functions on $\mathbb{R}^p$ analogous to the sectioning of sets $A \subseteq \mathbb{R}^p$ accomplished in 4.10. Thus this section complements the § 4 at the functional level. It belongs to the purely scalar part of the paper, and no vector measures appear in it. We shall adhere to the following notation:

$$(12.1)\qquad \begin{cases} p \in \mathbb{N}_+, \quad k \in [0,[p/2]], \quad \pi = \{\{i_1,j_1\},\dots,\{i_k,j_k\}\} \in \Pi_k^p, \\ M'_\pi := [1,p]\backslash M_\pi = \{m_1,\dots,m_{p-2k}\}, \quad m_1 < \dots < m_{p-2k}, \\ h = (h_1,h_2,\dots,h_{p-2k}) \in \mathbb{R}^{p-2k}. \end{cases}$$

We wish to get from the function f on $\mathbb{R}^p$ to the function $f_\pi^p(\cdot,h)$ on $\mathbb{R}^k$, by imitating the operations in § 4 which took us from the set $D \subseteq \mathbb{R}^p$ to the set $D_\pi^p(h) \subseteq \mathbb{R}^k$. That is, we hold the $m_1, m_2, \dots, m_{p-2k}$th variables fixed and equal to $h_1, h_2, \dots, h_{m-2k}$, respectively, and in the remaining $2k$ variables we identify the i_1th and j_1th, the i_2th and j_2th and so on, *à la* Wiener. We can formally define the function $f_\pi^p(\cdot,h)$ as follows:

12.2. *Definition.* Let p, k, π, h be as in (12.1). Then

(*a*) $\forall \tau \in \mathbb{R}^k$, $\theta_{\pi,h}^p(\tau) := t$, where $t = (t_1,\dots,t_p) \in \mathbb{R}^p$ is given by

$$\forall \alpha \in [1,k], \quad t_{i_\alpha} = \tau_\alpha = t_{j_\alpha} \quad \& \quad \forall \beta \in [1,p-2k], \quad t_{m_\beta} = h_\beta.$$

(*b*) For all functions f on $\mathbb{R}^p$, the π, h *section of* f is the function $f_\pi^p(\cdot,h)$ on $\mathbb{R}^k$ defined by

$$\forall \tau \in \mathbb{R}^k, \qquad f_\pi^p(\tau,h) := f\{\theta_{\pi,h}^p(\tau)\}.$$

(*c*) We define the operator $J_{\pi,h}^p$ by

$$\forall f \quad \text{on} \quad \mathbb{R}^p, \qquad (J_{\pi,h}^p)(f) := f \circ \theta_{\pi,h}^p = f_\pi^p(\cdot,h).$$

Note. $\theta_{\pi,h}^p(\tau)$ is the sole member of $\wp_{\pi^*}^{-1}(\tau) \cap I_\pi^p(h)$, which is a singleton, as the reader can check.

In $f_\pi^p(\tau,h)$ the components of τ and h can get thoroughly mixed as the following examples illustrate:

Example 1. Let $p = 11$, $k = 3$, $\tau \in \mathbb{R}^3$, $h \in \mathbb{R}^5$, &

$$\pi = \{\{3,4\},\{5,8\},\{6,10\}\} \in \Pi_3^{11}.$$

Then $\pi^* = \{4,8,10\}$ & $M'_\pi = \{1,2,7,9,11\}$. Hence $t := \theta_{\pi,h}^{11}(\tau)$ the sole member of $p_{\pi^*}^{-1}(\tau) \cap I_\pi^{11}(h)$ has the components

$$t_3 = t_4 = \tau_1, \qquad t_5 = t_8 = \tau_2, \qquad t_6 = t_{10} = \tau_3,$$

$$t_1 = h_1, \qquad t_2 = h_2, \qquad t_7 = h_3, \qquad t_9 = h_4, \qquad t_{11} = h_5.$$

Hence

$$t := \theta^{11}_{\pi,h}(\tau) = (h_1, h_2, \tau_1, \tau_1, \tau_2, \tau_3, h_3, \tau_2, h_4, \tau_3, h_5) \in \mathbb{R}^{11}.$$

For a function f on $\mathbb{R}^{11}$, $f^{11}_\pi(\tau, h)$ is the value of f at this t.

Example 2. Let $p = 16$, $k = 6$,

$$\pi = \{\{3,4\},\{5,8\},\{6,15\},\{7,9\},\{11,12\},\{13,14\}\} \in \Pi^{16}_6,$$

and so $M_\pi = [3,9] \cup [11,15] \subseteq [1,16]$ & $M'_\pi = \{1,2,10,16\}$.

Then, as can be checked, $\forall \tau \in \mathbb{R}^6$ & $\forall h \in \mathbb{R}^4$, $\wp^{-1}_{\pi^*}(\tau) \cap I^{16}_\pi(h) = \{t\}$, where

$$t := \theta^{16}_{\pi,h}(\tau) = (h_1, h_2, \tau_1, \tau_1, \tau_2, \tau_3, \tau_4, \tau_2, \tau_4, h_3, \tau_5, \tau_5, \tau_6, \tau_6, \tau_3, h_4).$$

Thus, $f^{16}_\pi(\tau, h)$, or more fully, $f^{16}_\pi(\tau_1 \cdots \tau_6; h_1 \cdots h_4)$, is the value of f at this point t.

However, a neat separation of the components of τ and h in $f^p_\pi(\tau, h)$ results when $\pi = \pi_k$, the k standard partition defined in (6.11), namely,

$$\pi_k := \{\{1,2\},\{3,4\} \cdots \{2k-1, 2k\}\} \in \Pi_{[1,2k]},$$

for which $M_\pi = [1, 2k]$ & $M'_\pi = [2k+1, p]$. We obviously have

(12.3) $$\forall \tau \in \mathbb{R}^k \quad \& \quad \forall h \in \mathbb{R}^{p-2k}, \quad f^p_{\pi_k}(\tau, h) = f(\tau_1, \tau_1, \ldots, \tau_k, \tau_k; h).$$

Further inquiry into the nature of such functional sectioning, requires a more detailed statement of the mapping $\theta^p_{\pi,h}$:

12.4. Triviality. *Let p, k, π, h be as in (2.1). Then*

(a) $\theta^p_{\pi,h}(\cdot)$ is a (single-valued) continuous function on $\mathbb{R}^k$ to $\mathbb{R}^p$;

(b) $\forall p \geqslant 2$ & $\forall k \in [1, [p/2]]$, Range $\theta^p_{\pi,h}(\cdot) \subseteq I^p_1 \subset \mathbb{R}^p$;

(c) $\forall p \in \mathbb{N}_+$ & $k = 0$, $\mathbb{R}^k = \{0\}$, $\pi = \emptyset$ & $\mathbb{R}^{p-2k} = \mathbb{R}^p$, $\theta^p_{\emptyset,h}(0) = h \in \mathbb{R}^p$, i.e. Range $\theta^p_{\emptyset,h}$ = the singleton $\{h\}$;

(d) for even p, & $k = p/2$, and $\mathbb{R}^{p-2k} = \mathbb{R}^0 = \{0\}$, and $\forall \tau \in \mathbb{R}^{p/2}$, $\theta^p_{\pi,h}(\tau) = t \in \mathbb{R}^p$, where $t_{i_\alpha} = \tau_\alpha = t_{j_\alpha}$, $\alpha \in [1, p/2]$;

(e) $\forall p \in \mathbb{N}_+$ & $k = 0$, $f^p_\pi(\tau, h) = f^p_\emptyset(0, h) = f(h)$;

(f) for even p, & $k = p/2$, $f^p_\pi(\tau, 0) = f(t)$, where t is as in (d).

Proof. (*a*) is obvious from definition 12.2(*a*).

(*b*) Let $p \geqslant 2$, $k \in [1, [p/2]]$ & $\pi = \{\{i_1, j_1\}, \ldots, \{i_k, j_k\}\} \in \Pi^p_k$. Then by 12.2(*a*), $\forall h \in \mathbb{R}^{p-2k}$ & $\forall \tau \in \mathbb{R}^k$, the i_αth and j_αth components of $t := \theta^p_{\pi,h}(\tau)$ are equal for all $\alpha \in [1, k]$. Therefore $\theta^p_{\pi,h}(\tau) \in I^p_1$. Thus (*b*).

(*c*) Let $p \in \mathbb{N}_+$ & $k = 0$. Then by (1.17), $\pi = \emptyset$. Also, $\mathbb{R}^k = \mathbb{R}^0 = \{0\}$ & $\mathbb{R}^{p-2k} = \mathbb{R}^p$. Hence $h \in \mathbb{R}^{p-2k} = \mathbb{R}^p$, and by definition 12.2(*a*), $\theta^p_{\emptyset,h}(0) = t$, where $\forall \beta \in [1, p]$, $t_\beta = h_\beta$, i.e. $t = h$. Thus (*c*).

(*d*) For even p and $k = p/2$, $M'_\pi = [1, p] \backslash M_\pi = \emptyset$, and the condition in 12.2(*a*) on $t := \theta^p_{\pi,h}(\tau)$ reduces to $\forall \alpha \in [1, p/2]$, $t_{i_\alpha} = \tau_\alpha = t_{j_\alpha}$. Thus (*d*).

(*e*), (*f*) These follow at once from (*c*) and (*d*) respectively, by virtue of the definition $f^p_\pi(\tau, h) = f\{\theta^p_{\pi,h}(\tau)\}$ in 12.2(*b*). ■

The rudimentary properties of the operator $J^p_{\pi,h}$ on the function-space $\mathbb{R}^{(\mathbb{R}^p)}$ into $\mathbb{R}^{(\mathbb{R}^k)}$ are stated in the next result, which is a functional analogue of the homomorphism proposition 4.11, and its corollary 4.12:

12.5. Proposition. *Let p, k, π, h be as in (12.1). Then*

(a) *$J^p_{\pi,h}$ is a linear and multiplicative on $\mathbb{R}^{(\mathbb{R}^p)}$ onto $\mathbb{R}^{(\mathbb{R}^k)}$, and preserves absolute value; more fully, we have $\forall f, g$ on $\mathbb{R}^p$, $\forall a, b \in \mathbb{R}$ & $\forall h \in \mathbb{R}^{p-2k}$,*

$$(af + bg)^p_\pi(\cdot, h) = af^p_\pi(\cdot, h) + bg^p_\pi(\cdot, h) \quad \text{on} \quad \mathbb{R}^k,$$
$$(f \cdot g)^p_\pi(\cdot, h) = f^p_\pi(\cdot, h) \cdot g^p_\pi(\cdot, h) \quad \text{on} \quad \mathbb{R}^k,$$
$$|f^p_\pi(\cdot, h)| = |f|^p_\pi(\cdot, h) \quad \text{on} \quad \mathbb{R}^k,$$

$\forall g$ on $\mathbb{R}^k$, $g \circ \wp_{\pi^*}$ is on $\mathbb{R}^p$ & $J^p_{\pi,h}(g \circ \wp_{\pi^*}) = g$ on $\mathbb{R}^k$;

(b) *$J^p_{\pi,h}$ is monotone, and continuous in the pointwise convergence topology, i.e.*

$$f(\cdot) \leqslant g(\cdot) \quad \text{on} \quad \mathbb{R}^p \quad \Longrightarrow \quad f^p_\pi(\cdot, h) \leqslant g^p_\pi(\cdot, h) \quad \text{on} \quad \mathbb{R}^k,$$
$$f_n(\cdot) \to f(\cdot) \quad \text{on} \quad \mathbb{R}^p \quad \Longrightarrow \quad (f_n)^p_\pi(\cdot, h) \to f^p_\pi(\cdot, h) \quad \text{on} \quad \mathbb{R}^k;$$

(c) *$J^p_{\pi,h}$ carries $\mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$ onto $\mathcal{M}(\mathcal{B}_k, \mathcal{B}_1)$;*

(d) *$\forall A \subseteq \mathbb{R}^p$, $[J^p_{\pi,h}(\chi_A)](\cdot) = \chi_{A^p_\pi(h)}(\cdot) = (\chi_A)^p_\pi(\cdot, h)$ on $\mathbb{R}^k$, and $J^p_{\pi,h}$ carries $\mathcal{S}(\mathcal{B}_p, \mathbb{R})$ onto $\mathcal{S}(\mathcal{B}_k, \mathbb{R})$;*

(e) *$\forall f$ on $\mathbb{R}^p$ and $\forall \phi \in \mathrm{Perm}(p)$,*

$$(f^{\phi^{-1}})^p_\pi(\cdot, h) = f^p_{\bar\pi}(\cdot, h^{\bar\phi}) \quad \text{on} \quad \mathbb{R}^k,$$

where $\bar\pi$ is the ϕ distortion of π, and $\bar\phi \in \mathrm{Perm}(p - 2k)$, is the (ϕ, π)-permutation of $[1, p - 2k]$, cf. definition 6.4;

(f) *$f \in \mathbb{R}^{(\mathbb{R}^p)}$ and $\operatorname{supp} f \subseteq \mathbb{R}^p_* := \mathbb{R}^p \backslash I^p_1 \Longrightarrow f \in$ Null space of $J^p_{\pi,h}$.*

Proof. (a), (b) From the fact that $\theta^p_{\pi,h}$ is a function on $\mathbb{R}^k$ to $\mathbb{R}^p$, and

(1) $$\forall f \in \mathbb{R}^{(\mathbb{R}^p)}, \qquad J^p_{\pi,h}(f) := f \circ \theta^p_{\pi,h},$$

all but one of the results in (a), (b) follow at once. Only the assertion in (a) that Range $J^p_{\pi,h} = \mathbb{R}^{(\mathbb{R}^p)}$ needs comment. Given g on $\mathbb{R}^k$ and π as in 12.1, we define f on $\mathbb{R}^p$ by $f = g \circ \wp_{\pi^*}$, i.e.

$$f(t) = f(t_1, t_2, \ldots, t_p) := g\{\wp_{\pi^*}(t)\} = g(t_{j_1}, t_{j_2}, \ldots, t_{j_k}),$$

where, cf. 12.1, $\{j_1, j_2, \ldots, j_k\} = \pi^*$. Then it is easily seen that $\forall h \in \mathbb{R}^{p-2k}$, $f^p_\pi(\cdot, h) = g(\cdot)$ on $\mathbb{R}^k$. Thus, $J^p_{\pi,h}(f) = g$.

(c) Since $\theta^p_{\pi,h}$ is continuous on $\mathbb{R}^k$ to $\mathbb{R}^p$, therefore $\theta^p_{\pi,h} \in \mathcal{M}(\mathcal{B}_k, \mathcal{B}_p)$. Hence if $f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$, then from (1), $J^p_{\pi,h}(f) \in \mathcal{M}(\mathcal{B}_k, \mathcal{B}_1)$. Thus ($c$).

(d) As the reader can easily check,

(2) $$t = \theta^p_{\pi,h}(\tau) \quad \text{iff} \quad \{\tau\} = \wp_{\pi^*}[\{t\} \cap I^p_\pi(h)].$$

It follows from this that for any $A \subseteq \mathbb{R}^p$, $(\chi_A)^p_\pi(\tau, h) = 1$ iff $\chi_{A^p_\pi(h)}(\tau) = 1$, and therefore that $(\chi_A)^p_\pi(\cdot, h) = \chi_{A^p_\pi(h)}(\cdot)$ on $\mathbb{R}^k$. It so follows from the linearity of $J^p_{\pi,h}$ that it carries $\mathcal{S}(\mathcal{B}_p, \mathbb{R})$ into $\mathcal{S}(\mathcal{B}_k, \mathbb{R})$. Thus ($d$).

(e) Let $\tau \in \mathbb{R}^k$ and $s := \theta_{\pi,h}(\tau)$, so that

(3) $$s_{i_\alpha} = \tau_\alpha = s_{j_\alpha} \quad \& \quad s_{m_\beta} = h_\beta, \quad \alpha \in [1, k], \quad \beta = [1, p - 2k].$$

Then letting $\forall n \in [1, p]$, $t_n := s_{\phi^{-1}(n)}$, we have

(4) $$(f^{\phi^{-1}})^p_\pi(\tau, h) := f^{\phi^{-1}}(s) := f(s_{\phi^{-1}(1)}, \ldots, s_{\phi^{-1}(p)}) = f(t_1, \ldots, t_p).$$

The arguments on the RHS form a mixture of τ_α's and h_β's. However, for $\alpha \in [1,k]$ and $\beta \in [1, p-2k]$, we see from (3) that

$$(5)\qquad \begin{cases} t_{\phi(i_\alpha)} = s_{\phi^{-1}\{\phi(i_\alpha)\}} = s_{i_\alpha} = \tau_\alpha = s_{j_\alpha} = s_{\phi^{-1}\{\phi(j_\alpha)\}} = t_{\{\phi(j_\alpha)\}}, \\ t_{\phi(m_\beta)} = s_{\phi^{-1}\{\phi(m_\beta)\}} = s_{m_\beta} = h_\beta. \end{cases}$$

Since the cells $\{\bar\imath_\alpha, \bar\jmath_\alpha\}$ of $\bar\pi$ are defined by

$$\bar\imath_\alpha = \phi(i_\alpha) \wedge \phi(j_\alpha), \qquad \bar\jmath_\alpha = \phi(i_\alpha) \vee \phi(j_\alpha),$$

therefore $\{\bar\imath_\alpha, \bar\jmath_\alpha\} = \{\phi(i_\alpha), \phi(j_\alpha)\}$, and it follows that $t_{\bar\imath_\alpha}$ and $t_{\bar\jmath_\alpha}$ are equal to $t_{\phi(i_\alpha)} = \tau_\alpha$, by (5). Also from the definition of $\bar\phi$, $\bar m_\beta := \phi\{m_{\bar\phi(\beta)}\}$, and hence $t_{\bar m_\beta} = t_{\phi\{m_{\bar\phi(\beta)}\}} = h_{\bar\phi(\beta)} = (h^{\bar\phi})_\beta$, by (5). Thus we have

$$t_{\bar\imath_\alpha} = \tau_\alpha = t_{\bar\jmath_\alpha} \quad \& \quad t_{\bar m_\beta} = (h^{\bar\phi})_\beta, \quad \alpha \in [1,k], \quad \beta \in [1, p-2k],$$

i.e. by 12.2, $(t_1, \dots, t_p) = \theta^p_{\bar\pi, h^{\bar\phi}}(\tau)$. Thus (4) reduces to

$$(f^{\phi^{-1}})^p_\pi(\tau, h) = f\{\theta^p_{\bar\pi, h^{\bar\phi}}(\tau)\} =: f^p_{\bar\pi}(\tau, h^{\bar\phi}).$$

Thus (e).

(f) Let $p \geqslant 2$ and $k \in [1, [p/2]]$. Then by 12.4(b), Range $\theta^p_{\pi,h}(\cdot) \subseteq I^p_1$. Now let supp $f \subseteq \mathbb{R}^p_* := \mathbb{R}^p \backslash I^p_1$. Then $I^p_1 \subseteq$ Null space f. Thus Range $\theta^p_{\pi,h}(\cdot) \subseteq$ Null space f. Hence $J^p_{\pi,h}(f) = f \circ \theta^p_{\pi,h} = 0$. Thus (f). ∎

Note. We can see from 12.5(f) that the operator $J^p_{\pi,h}$ is not one–one. The easiest example, for $p = 2$, is offered by $f := \chi_{\mathbb{R}^2 \backslash I}$ on $\mathbb{R}^2$, where I is the diagonal of $\mathbb{R}^2$.

The equality in 12.5(d) establishes the nexus between functional and set-theoretical sectioning. It reduces when A is an interval to:

$$\forall P \in \mathcal{P}_p, \quad \forall \tau \in \mathbb{R}^k \quad \& \quad \forall h \in \mathbb{R}^{p-2k}, \quad (\chi_P)^p_\pi(\tau, h) = \chi_{P(\pi)}(\tau) \cdot \chi_{P_{M'_\pi}}(h),$$

by 4.10 and 4.9(b). The equality 12.5(d) immediately yields an expression for the canonical coefficients, cf. 4.13 and 4.18, in terms of the sectioning of indicator functions:

$$(12.6)\qquad \begin{cases} \text{With the notation 12.1, } D \in \mathcal{D}_p,\ B \in \mathcal{B}_p \ \&\ h \in \mathbb{R}^{p-2k}, \text{ we have} \\ (a) \quad \lambda^p_\pi(D,h) = \ell_k\{D^p_\pi(h)\} = \displaystyle\int_{\mathbb{R}^k} (\chi_D)^p_\pi(\tau,h)\ell_k\,(\mathrm{d}\tau), \\ (b) \quad \gamma^p_k(D,h) = \displaystyle\int_{\mathbb{R}^k} \left[\sum_{\pi \in \Pi^p_k} (\chi_D)^p_\pi(\tau,h)\right] \ell_k\,(\mathrm{d}\tau), \\ (c) \quad |\lambda^p_\pi|(B,h) := |\ell_k|\{B^p_\pi(h)\} = \displaystyle\int_{\mathbb{R}^k} (\chi_B)^p_\pi(\tau,h)\ell_k\,(\mathrm{d}\tau). \end{cases}$$

The equality in (12.6)(a) can of course be written:

$$(12.7)\qquad \int_{\mathbb{R}^p} (\chi_D)(t)\lambda^p_\pi\,(\mathrm{d}t, h) = \int_{\mathbb{R}^k} (\chi_D)^p_\pi(\tau,h)\ell_k\,(\mathrm{d}\tau).$$

12.8. Heuristics. The ground to be covered in the rest of this section is determined largely by the following heuristic rule:

Rule. Results for $\mathbb{E}_{\xi_p}(f)$*, where* f *is on* $\mathbb{R}^p$*, can be conjectured from those for*

$\xi_p(D)$, for $D \in \mathcal{D}_p$ given in §§ 3–11, by replacing all $\rho(C)$, where ρ is a measure, by $\mathbb{E}_\rho(\chi_C)$, then replacing χ_D by f, and χ_C by an appropriate transform of f.

For instance, the result in 11.10(a):

$$\xi_p(D) = \sum_{k=0}^{[p/2]} \int_{\mathbb{R}^{p-2k}} \gamma_k^p(D,h)\eta_{p-2k}(\mathrm{d}h)$$

can by virtue of (12.6)(b) be transcribed as

$$\mathbb{E}_{\xi_p}(\chi_D) = \sum_{k=0}^{[p/2]} \int_{\mathbb{R}^{p-2k}} \left\{ \int_{\mathbb{R}^k} \left[\sum_{\pi \in \Pi_k^p} (\chi_D)_\pi^p(\tau,h) \right] \ell_k(\mathrm{d}\tau) \right\} \eta_{p-2k}(\mathrm{d}h),$$

and this suggests the general formula for f (in place of χ_D):

(1) $$\mathbb{E}_{\xi_p}(f) = \sum_{k=0}^{[p/2]} \int_{\mathbb{R}^{p-2k}} \left\{ \int_{\mathbb{R}^k} \left[\sum_{\pi \in \Pi_k^p} f_\pi^p(\tau,h) \right] \ell_k(\mathrm{d}\tau) \right\} \eta_{p-2k}(\mathrm{d}h).$$

This formula is valid, cf. 13.12 and 13.11(b), below. However, to demonstrate it, we will have to appeal to a Fubini-type theorem for Markovian kernels $K(D,h)$ resembling $\gamma_k^p(D,h)$, which is established in Appendix B.

As far as this section is concerned, the Rule suggests a study of the ℓ_k-integrability of

$$\sum_{\pi \in \Pi_\pi^p} f_\pi^p(\cdot,h)$$

on $\mathbb{R}^k$. This is carried out in corollary 12.10, after the preliminary proposition devoted to individual $f_\pi^p(\cdot,h)$.

12.9. Proposition. *Let* (i) p, k, π, h *be as in (12.1),* (ii) $f \in \mathcal{M}(\mathcal{B}_p,\mathcal{B}_1)$. *Then*

(a)

$$\int_{\mathbb{R}^p} |f(t)| \cdot |\lambda_\pi^p|(\mathrm{d}t,h) = \int_{\mathbb{R}^k} |f_\pi^p(\tau,h)| \cdot |\ell_k|(\mathrm{d}t);$$

(b) *the restriction of $J_{\pi,h}^p$ to $L_{1,\lambda_\pi^p(\cdot,h)}$ is a linear isometry on $L_{1,\lambda_\pi^p(\cdot,h)}$ onto $L_1(\mathbb{R}^k)$;*

(c) $\forall f \in L_{1,\lambda_\pi^p(\cdot,h)}$,

$$\int_{\mathbb{R}^p} f(t)\lambda_\pi^p(\mathrm{d}t,h) = \int_{\mathbb{R}^k} f_\pi^p(\tau,h)\ell_k(\mathrm{d}t);$$

(d) $H_\pi^p(f) := \{h : h \in \mathbb{R}^{p-2k} \ \&\ f_\pi^p(\cdot,h) \in L_1(\mathbb{R}^k)\} \in \mathcal{B}_{p-2k}$;

(e) $\forall \phi \in \mathrm{Perm}(p)$, $H_\pi^p(f^\phi) = \{H_{\bar\pi}^p(f)\}^{\psi^{-1}}$, *where $\bar\pi \in \Pi_k^p$ is the ϕ^{-1} distortion of π, and ψ is the (ϕ^{-1},π) permutation of* $[1,p-2k]$, *cf. definition 6.4.*

Proof. (a) Write for brevity, $\lambda(\cdot) = \lambda_\pi^p(\cdot,h)$ and $J = J_{\pi,h}^p$. Then since by 12.5(a), $|J(f)| = J(|f|)$, to prove (a) we have only to show that

(I) $$\mathbb{E}_{|\lambda|}(|f|) = \mathbb{E}_{|\ell_k|}\{J(|f|)\} \in [0,\infty].$$

Proof of (I). For $f = \chi_A$, $A \in \mathcal{B}_p$, $J(|f|) = \chi_{A_\pi^p(h)}$ by 12.5(d), and (I) reduces to the first equality in (12.6)(c). It follows in turn from the linearity of $\mathbb{E}_{|\lambda|}$ and of $\mathbb{E}_{|\ell_k|} \circ J$, and 12.5($d$) that

(1) $$\forall f \in \mathcal{S}(\mathcal{B}_p,\mathbb{R}), \qquad \text{(I) holds.}$$

Finally, let $f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$ and let $f_n \in S(\mathcal{B}_p, \mathbb{R})$ be such that as $n \to \infty$,

$$(2) \qquad f_n(\cdot) \to f(\cdot) \quad \& \quad |f_n(\cdot)| \uparrow |f(\cdot)| \quad \text{on} \quad \mathbb{R}^p.$$

Then by 12.5(*b*), $J(|f_n|) \uparrow J(|f|)$ on $\mathbb{R}^k$. Hence by (1) and two applications of the monotone convergence theorem, we see that (I) holds for f. Thus (*a*).

(*b*) This is immediate from (*a*).

(*c*) By (12.7), (*c*) holds for $f = \chi_D$, for $D \in \mathcal{D}_p$, and thence for $f \in S(\mathcal{D}_p, \mathbb{R})$. Now let $f \in L_{1,\lambda}$. Then by (*b*), $J(f) \in L_1(\mathbb{R}^k)$. Letting $f_n(\cdot)$ be as in (2), we readily infer from Lebesgue's dominated convergence theorem that (*c*) holds for f. Thus (*c*).

(*d*) $\forall(\tau, h) \in \mathbb{R}^k \times \mathbb{R}^{p-2k}$, write $\bar{\theta}(\tau, h) = \theta^p_{\pi,h}(\tau)$. Then from 12.3(*a*), we see that $\bar{\theta}$ is a continuous function on $\mathbb{R}^{p-k}$ to $\mathbb{R}^p$. Since $f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$, it follows that

$$(3) \qquad f^p_\pi(\cdot,\cdot) := (f \circ \bar{\theta})(\cdot,\cdot) \in \mathcal{M}(\mathcal{B}^{p-k}, \mathcal{B}_1).$$

Hence by Tonelli's theorem, the partial integral

$$F(\cdot) := \int_{\mathbb{R}^k} |f^p_\pi(\tau,\cdot)||\ell_k|\,(\mathrm{d}\tau) \in \mathcal{M}(\mathcal{B}_{p-2k}, \mathcal{B}_{[0,\infty]}),$$

where $\mathcal{B}_{[0,\infty]}$ is the Borel σ-algebra over $\mathbb{R} \cup \{\infty\}$. Hence

$$H^p_\pi(f) = \{h : h \in \mathbb{R}^{p-2k} \ \& \ F(h) < \infty\} \in \mathcal{B}_{p-2k}.$$

Thus (*d*).

(*e*) Let $\phi \in \mathrm{Perm}(p)$ and let $\bar{\pi}$ and ψ be as indicated. Then taking ϕ^{-1} instead of ϕ in 12.5(*e*), we see that $\forall h \in \mathbb{R}^{p-2k}$,

$$\begin{aligned} h \in H^p_\pi(f^\phi) \quad &\Longleftrightarrow \quad (f^\phi)^p_\pi(\cdot, h) \in L_1(\mathbb{R}^k) \quad \Longleftrightarrow \quad f^p_{\bar{\pi}}(\cdot, h^\psi) \in L_1(\mathbb{R}^k) \\ &\Longleftrightarrow \quad h^\psi \in H^p_{\bar{\pi}}(f) \quad \Longleftrightarrow \quad h \in \{H^p_{\bar{\pi}}(f)\}^{\psi^{-1}}. \end{aligned}$$

Thus (*e*). ∎

The analogue of 12.9, in which the kernels $\lambda^p_\pi(\cdot,\cdot)$ are replaced by the kernels $\gamma^p_k(\cdot,\cdot)$ reads as follows:

12.10. Corollary. *Let* (i) p, k, h *be as in (12.1),* (ii) $f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$. *Then*

(*a*)

$$\int_{\mathbb{R}^p} |f(t)| \cdot |\gamma^p_k|\,(\mathrm{d}t, h) = \int_{\mathbb{R}^k} \Big\{ \sum_{\pi \in \Pi^p_k} |f^p_\pi(\tau, h)| \Big\} |\ell_k|\,(\mathrm{d}\tau) \in [0, \infty];$$

(*b*)

$$f \in L_{1,\gamma^p_k(\cdot,h)} \quad \Longleftrightarrow \quad \sum_{\pi \in \Pi^p_k} |f^p_\pi(\cdot, h)| \in L_1(\mathbb{R}^k);$$

(*c*) $\forall f \in L_{1,\gamma^p_k(\cdot,h)}$,

$$\int_{\mathbb{R}^p} f(t)\gamma^p_k\,(\mathrm{d}t, h) = \int_{\mathbb{R}^k} \Big\{ \sum_{\pi \in \Pi^p_k} f^p_\pi(\tau, h) \Big\} \ell_k\,(\mathrm{d}\tau).$$

Proof. (*a*) Recalling that by 4.18(*c*),

$$(1) \qquad |\gamma^p_k|(\cdot, h) := \sum_{\pi \in \Pi^p_k} |\lambda^p_\pi|(\cdot, h) \quad \text{on} \quad \mathcal{B}_p,$$

we see from (1) and 12.9(a) that

$$\int_{\mathbb{R}^p} |f(t)| \cdot |\gamma_k^p| \,(\mathrm{d}t, h) = \sum_{\pi\in\Pi_k^p} \int_{\mathbb{R}^p} |f(t)| \cdot |\lambda_\pi^p| \,(\mathrm{d}t, h)$$
$$= \sum_{\pi\in\Pi_k^p} \int_{\mathbb{R}^k} |f_\pi^p(\tau, h)| \cdot |\ell_k| \,(\mathrm{d}\tau)$$
$$= \int_{\mathbb{R}^k} \Big\{ \sum_{\pi\in\Pi_k^p} |f_\pi^p(\tau, h)| \Big\} |\ell_k| \,(\mathrm{d}\tau).$$

Thus (a).

(b) follows once more from (a).

(c) Let $f \in L_{1,\gamma_k^p(\cdot,h)}$. Then by ($b$), $\sum_{\pi\in\Pi_k^p} f_\pi^p(\cdot, h) \in L_1(\mathbb{R}^k)$. Hence by the definition 4.13 of $\gamma_k^p(\cdot,\cdot)$,

$$\int_{\mathbb{R}^p} f(t)\gamma_k^p \,(\mathrm{d}t, h) = \int_{\mathbb{R}^p} f(t) \sum_{\pi\in\Pi_k^p} \lambda_\pi^p \,(\mathrm{d}t, h) = \sum_{\pi\in\Pi_k^p} \int_{\mathbb{R}^p} f(t)\lambda_\pi^p \,(\mathrm{d}t, h)$$
$$= \sum_{\pi\in\Pi_k^p} \int_{\mathbb{R}^p} f_\pi^p(\tau, h)\ell_k \,(\mathrm{d}\tau), \qquad \text{by 12.9}(c)$$
$$= \int_{\mathbb{R}^p} \sum_{\pi\in\Pi_k^p} f_\pi^p(\tau, h)\ell_k \,(\mathrm{d}\tau).$$

Thus (c). ■

The heuristically obtained equation 12.8(1) indicates that not only do we want the sum $\sum_{\pi\in\Pi_k^p} f_\pi^p(\cdot, h)$ to be in $L_1(\mathbb{R}^k)$, but want the h for which this holds to form a carrier of ℓ_{p-2k}, so that we can subject $\mathbb{E}_{\ell_k}\{\sum_{\pi\in\Pi_k^p} f_\pi^p(\cdot, h)\}$ to a further integration with respect to ℓ_{p-2k}. These requirements can be conceptualized as follows:

12.11. *Definition.* ($p - 2k$ marginalization) Let $p \in \mathbb{N}_+$ and $k \in [0, [p/2]]$. Then

(a) $\forall f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$ & $\forall \pi \in \Pi_k^p$,

$$H_k^p(f) := \{h : h \in \mathbb{R}^{p-2k} \ \&\ f \in L_{1,\gamma_k^p(\cdot,h)}\}.$$

(b) $\mathcal{M}_k^p := \{f : f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1) \ \&\ H_k^p(f)$ is a carrier of $\ell_{p-2k}\}$.

(c) $\forall f \in \mathcal{M}_k^p$,

$$M_k^p(f) := f_k^p(\cdot) := \chi_{H_k^p(f)}(\cdot) \int_{\mathbb{R}^p} f(t)\gamma_k^p \,(\mathrm{d}t, \cdot) \quad \text{on} \quad \mathbb{R}^{p-2k}.$$

(d) The function $f_k^p(\cdot)$ on $\mathbb{R}^{p-2k}$ is called the $p - 2k$ *marginalization of* f, and M_k^p is called the $p - 2k$ *marginalization operator.*

Our immediate objective is to show that $\mathcal{M}_k^p$ is a vector space closed under permutations, and then to investigate the linearity and range of the operator M_k^p on $\mathcal{M}_k^p$. These rest on the following simple properties of the sets $H_k^p(f)$, the proof of which we leave to the reader.

12.12. Triviality. *Let* (i) $p \in \mathbb{N}_+$, $k \in [0, [p/2]]$, (ii) $f, g \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$. *Then*

(a) $H_k^p(f) \in \mathcal{B}_{p-2k}$; *for* $f = \chi_D$, $D \in \mathcal{D}_p$, $H_k^p(\chi_D) = \mathbb{R}^{p-2k}$;

(*b*)

$$H_k^p(f) = \bigcap_{\pi\in\Pi_k^p} H_\pi^p(f) = \left\{h : h \in \mathbb{R}^{p-2k} \ \& \sum_{\pi\in\Pi_k^p} f_\pi^p(\cdot,h) \in L_1(\mathbb{R}^k)\right\},$$

cf. 12.9(*d*);

(*c*) $H_k^p(|f|) = H_k^p(f)$;
(*d*) $H_k^p(cf) = H_k^p(f)$, $c \in \mathbb{R}\backslash\{0\}$, $H_k^p(0\cdot f) = \mathbb{R}^{p-2k}$;
(*e*) $H_k^p(f) \cap H_k^p(g) \subseteq H_k^p(f+g)$;
(*f*) $|f(\cdot)| \leqslant |g(\cdot)|$ *on* $\mathbb{R}^p \Longrightarrow H_k^p(g) \subseteq H_k^p(f)$;
(*g*) $H_0^p(f) = \mathbb{R}^p$;
(*h*) *for even p*,

$$H_{p/2}^p(f) = \{0\} \quad \textit{iff} \quad \forall\pi \in \Pi_{p/2}^p, \quad f_\pi^p(\cdot,0) \in L_2(\mathbb{R}^{p/2})$$
$$H_{p/2}^p(f) = \emptyset \quad \textit{iff} \quad \exists\pi \in \Pi_{p/2}^p \ni f_\pi^p(\cdot,0) \notin L_2(\mathbb{R}^{p/2}).$$

That the inclusion in (*e*) can be proper is seen on taking any f in (ii) and $g = -f$, and noting that by (*d*), the RHS $= \mathbb{R}^{p-2k}$.

12.13. Theorem. (The properties of $\mathcal{M}_k^p$) *Let $p \in \mathbb{N}_+$ and $k \in [0,[p/2]]$. Then*
(*a*) *$\mathcal{M}_k^p$ is a linear submanifold of $\mathcal{M}(\mathcal{B}_p,\mathcal{B}_1)$, which is closed under absolute valuation and under the permutation group* Perm(p);
(*b*) $f \in \mathcal{M}(\mathcal{B}_p,\mathcal{B}_1)$ & $|f(\cdot)| \leqslant \phi(\cdot) \in \mathcal{M}_k^p \Longrightarrow f \in \mathcal{M}_k^p$;
(*c*) $\mathcal{S}(\mathcal{D}_p,\mathbb{R}) \subseteq \mathcal{M}_k^p$;
(*d*)

$$\mathcal{M}_k^p = \left\{f : f \in \mathcal{M}(\mathcal{B}_p,\mathcal{B}_1) \ \& \ \exists N_f \in \mathbb{R}^p \ni \forall h \in \mathbb{R}^{p-2k}\backslash N_f, \sum_{\pi\in\Pi_k^p} f_\pi^p(\cdot,h) \in L_1(\mathbb{R}^k)\right\};$$

(*e*) $\mathcal{M}_0^p = \mathcal{M}(\mathcal{B}_p,\mathcal{B}_1)$, *and for even p*,

$$\mathcal{M}_{p/2}^p = \left\{f : f \in \mathcal{M}(\mathcal{B}_p,\mathcal{B}_1) \ \& \sum_{\pi\in\Pi_{p/2}^p} f_\pi^p(\cdot,0) \in L_1(\mathbb{R}^{p/2})\right\}.$$

Proof. (*a*) From 12.12(*b*), (*c*), (*d*), we see that if $H_k^p(f)$, $H_k^p(g)$ are carriers of ℓ_{p-2k}, then so are $H_k^p(|f|)$ and $H_k^p(af+bg)$, $a,b \in \mathbb{R}$. This shows the $\mathcal{M}_k^p$ is linear and closed under absolute valuation. It only remains to show that

(I) $$f \in \mathcal{M}_k^p \quad \& \quad \phi \in \text{Perm}(p) \quad \Longrightarrow \quad f^\phi \in \mathcal{M}_k^p.$$

Proof of (I). Let $f \in \mathcal{M}_k^p$, $\phi \in \text{Perm}(p)$, $\pi \in \Pi_k^p$ and let $\bar\pi$ and ψ be as in 12.9(*e*). Then by 12.12(*b*),

$$H_{\bar\pi}^p(f) \supseteq H_k^p(f) = \text{ a carrier of } \ell_{p-2k}.$$

Thus $H_{\bar\pi}^p(f)$ is a carrier of ℓ_{p-2k}, and by the ψ invariance of ℓ_{p-2k}, so is $\{H_{\bar\pi}^p(f)\}^{\psi^{-1}} = H_k^p(f^\phi)$, by 12.9(*e*). This holds $\forall\pi \in \Pi_k^p$. Hence, $H_k^p(f^\phi) = \bigcap_{\pi\in\Pi_k^p} H_\pi^p(f^\phi)$ is also a carrier of ℓ_{p-2k}. Thus $f^\phi \in \mathcal{M}_k^p$ and (I) and (*a*) are proved.

(*b*) If $\phi \in \mathcal{M}_k^p$, then $H_k^p(\phi)$ is a carrier. Hence, by 12.12(*f*), so is $H_k^p(f)$, i.e. $f \in \mathcal{M}_k^p$. Thus (*b*).

(*c*) First, for $D \in \mathcal{D}_p$, by 12.12(*a*), $H_k^p(\chi_D) = \mathbb{R}^{p-2k}$, and therefore $\chi_D \in \mathcal{M}_k^p$. By the linearity of $\mathcal{M}_k^p$, we have (*c*).

(d) A function $f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$ will be in the set on the RHS of (d) iff the set

$$\left\{h : h \in \mathbb{R}^{p-2k} \ \& \sum_{\pi \in \Pi_k^p} f_\pi^p(\cdot, h) \in L_1(\mathbb{R}^k)\right\}$$

is a carrier of ℓ_{p-2k}. But by 12.12(b), the last set is $H_k^p(f)$. Thus (d) just restates the definition 12.11(b) of $\mathcal{M}_k^p$. Thus (d).

(e) Let $f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$. For $k = 0$, $H_0^p(f) = \mathbb{R}^p$ by 12.12(g), and hence $f \in \mathcal{M}_0^p$. Thus, $\mathcal{M}_0^p = \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$. Next, for p even and $k = p/2$, f will be in $\mathcal{M}_{p/2}^p$ if and only if $H_{p/2}^p(f)$ is a carrier of the measure $\ell_{p-2k} = \ell_0$ over $\mathbb{R}^0 = \{0\}$, i.e. if and only if $H_{p/2}^p(f) = \{0\}$. But by 12.12(h), this is the case if and only if $\forall \pi \in \Pi_{p/2}^p$, $f_{p/2}^p(\cdot, 0) \in L_1(\mathbb{R}^{p/2})$. Thus, $\mathcal{M}_{p/2}^p$ is as stated. Thus (e). ∎

The corresponding proposition for the marginalization operator M_k^p reads as follows:

12.14. Theorem. (The properties of M_k^p) *Let $p \in \mathbb{N}_+$ and $k \in [0, [p/2]]$. Then*

(a) $\forall f \in \mathcal{M}_k^p$,

$$f_k^p(\cdot) := [M_k^p(f)](\cdot) = \chi_{H_k^p(f)}(\cdot) \int_{\mathbb{R}^k} \left\{\sum_{\pi \in \Pi_k^p} f_\pi^p(\tau, \cdot)\right\} \ell_k(\mathrm{d}\tau) \quad on \quad \mathbb{R}^{p-2k};$$

(b) *the operator M_k^p is on $\mathcal{M}_k^p$ to $\mathcal{M}(\mathcal{B}_{p-2k}, \mathcal{B}_1)$, and M_k^p is linear and monotone, 'mod ℓ_{p-2k}', i.e. $\forall f, g \in \mathcal{M}_k^p$ & $\forall a, b \in \mathbb{R}$,*

$$M_k^p(af + bg) = aM_k^p(f) + bM_k^p(g), \quad \text{a.e.} \quad \ell_{p-2k} \text{ on } \mathbb{R}^{p-2k}$$

&

$$f \leqslant g \text{ on } \mathbb{R} \Longrightarrow M_k^p(f) \leqslant M_k^p(g), \quad \text{a.e.} \quad \ell_{p-2k} \quad \text{on} \quad \mathbb{R}^{p-2k};$$

(c) *if $\forall n \in \mathbb{N}_+$, $f_n \in \mathcal{M}_k^p$ & $|f_n(\cdot)| \leqslant F(\cdot) \in \mathcal{M}_k^p$, and $f_n(\cdot) \to f(\cdot)$ on $\mathbb{R}^p$, then*

$$M_k^p(f_n) \to M_k^p(f) \quad \text{on} \quad H_k^p(F), \quad \text{as} \quad n \to \infty;$$

(d) $\forall D \in \mathcal{D}_p$, $[M_k^p(\chi_D)](\cdot) = (\chi_D)_k^p(\cdot) = \gamma_k^p(D, \cdot)$ *on* $\mathbb{R}^p$;

(e) $\forall p \geqslant 2$, $\forall k \in [1, [p/2]]$, & $\forall f \in \mathcal{M}_k^p$,

$$\operatorname{supp} f \subseteq \mathbb{R}_*^p := \mathbb{R}^p \backslash I_1^p \Longrightarrow f \in \text{ Null space of } M_k^p;$$

(f) *for $k = 0$, we have $\forall f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$, $M_0^p(f) = f_0^p = f$ on $\mathbb{R}^p$; for even p, $\forall f \in \mathcal{M}_{p/2}^p$,*

$$[M_{p/2}^p(f)](0) = f_{p/2}^p(0) = \int_{\mathbb{R}^p} f(t) \gamma_{p/2}^p(\mathrm{d}t, 0).$$

Proof. (a) Let $f \in \mathcal{M}_k^p$. Then by definition 12.11(c) and corollary 12.10(c),

$$M_k^p(f)(\cdot) =: \chi_{H_k^p(f)}(\cdot) \int_{\mathbb{R}^p} f(t) \gamma_k^p(\mathrm{d}t, \cdot) = \chi_{H_k^p(f)}(\cdot) \int_{\mathbb{R}^k} \sum_{\pi \in \Pi_k^p} f_k^p(\tau, \cdot) \ell_k(\mathrm{d}\tau).$$

Thus (a).

(b) Let $f, g \in \mathcal{M}_k^p$ & $a, b \in \mathbb{R}$. Then $\forall h \in \mathbb{R}^{p-2k}$,

$$\begin{aligned} [M_k^p(af + bg)](h) &= \chi_{H_k^p(af+bg)}(h) \cdot \mathbb{E}_{\gamma_k^p(\cdot, h)}(af + bg) \\ &= \chi_{H_k^p(af+bg)}(h)[a\mathbb{E}_{\gamma_k^p(\cdot, h)}(f) + b\mathbb{E}_{\gamma_k^p(\cdot, h)}(g)]. \end{aligned} \quad (1)$$

But since f, g, and by 12.13(a), $f+g$ are in $\mathcal{M}_k^p$, the sets $H_k(f)$, $H_k(g)$ and $H_k(af+bg)$ are carriers of ℓ_{p-2k}, and their indicator functions are equal, a.e. ℓ_{p-2k}. Thus for ℓ_{p-2k} almost all $h \in \mathbb{R}^{p-2k}$, the indicator on the RHS(1) is, cf. 12.12(d), replaceable by the indicators of $H_k(f)$, $H_k(g)$, and so

$$\begin{aligned}\text{RHS}(1) &= \chi_{H_k^p(f)} a \mathbb{E}_{\gamma_k^p(\cdot,h)}(f) + \chi_{H_k^p(g)} b \mathbb{E}_{\gamma_k^p(\cdot,h)}(g) \\ &= a[M_k^p(f)](h) + b[M_k^p(g)](h). \end{aligned} \tag{2}$$

Likewise, if $f \leqslant g$ on $\mathbb{R}^p$, then for all $h \in \mathbb{R}^{p-2k}$, $\mathbb{E}_{\gamma_k^p(\cdot,h)}(f) \leqslant \mathbb{E}_{\gamma_k^p(\cdot,h)}(g)$, and so for ℓ_{p-2k} almost all h,

$$\chi_{H_k^p(f)}(h)\mathbb{E}_{\gamma_k^p(\cdot,h)}(f) \leqslant \chi_{H_k^p(g)}(h)\mathbb{E}_{\gamma_k^p(\cdot,h)}(g). \tag{3}$$

By (1), (2), (3) we have (b).

(c) By 12.13(b), all f_n and f are in $\mathcal{M}_k^p$. Now let $h \in H_k^p(F)$. Then $|f_n(\cdot)| \leqslant F(\cdot) \in L_{1,\gamma_k^p(\cdot,h)}$. From the datum $f_n \to f$ on $\mathbb{R}^p$ and Lebesgue's dominated convergence theorem, it thus follows that

$$\mathbb{E}_{\gamma_k^p(\cdot,h)}(f_n) \to \mathbb{E}_{\gamma_k^p(\cdot,h)}(f). \tag{4}$$

Since $h \in H_k^p(F)$, therefore by 12.12(f), h is in $H_k^p(f_n)$ and $H_k^p(f)$, i.e. $\chi_{H_k^p(f_n)}(h) = 1 = \chi_{H_k^p(f)}$. And from (4) we get $M_k^p(f_n) \to M_k^p(f_n)$ on $H_k^p(F)$. Thus (c).

(d) Let $D \in \mathcal{D}_p$. Then by 12.12(a), $H_k^p(\chi_D) = \mathbb{R}^{p-2k}$. Hence by 12.11($c$), $\forall h \in \mathbb{R}^{p-2k}$,

$$M_k^p(\chi_D)(h) := (\chi_D)_k^p(h) := \int_{\mathbb{R}^p} \chi_D(t)\gamma_k^p(\mathrm{d}t, h) = \gamma_k^p(D, h).$$

Thus (d).

(e) Let $p \geqslant 2$, $k \in [0, [p/2]]$ and $f \in \mathcal{M}_k^p$ & $\operatorname{supp} f \subseteq \mathbb{R}_*^p$. Then by 12.5($f$),

$$\forall \pi \in \Pi_k^p \quad \& \quad \forall h \in \mathbb{R}^{p-2k}, \qquad J_{\pi,h}^p(f) = 0. \tag{5}$$

But by (a) and the definition 12.2(c) of $J_{\pi,h}^p$, $\forall h \in \mathbb{R}^{p-2k}$,

$$[M_k^p(f)](h) = \chi_{H_k^p(f)}(h) \cdot \mathbb{E}_{\ell_k}\Bigg\{\sum_{\pi \in \Pi_k^p} J_{\pi,h}(f)\Bigg\} = 0, \qquad \text{by (5)}.$$

Thus $M_k^p(f) = 0$. Thus (e).

(f) For $k = 0$, we have $\mathcal{M}_0^p = \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$, by 12.13($e$). And by 12.12($g$), $H_0^p(f) = \mathbb{R}^p$. Hence by 12.11(c),

$$\forall h \in \mathbb{R}^{p-2k}, \qquad [M_0^p(f)](h) := f_0^p(h) := \int_{\mathbb{R}^p} f(t)\gamma_0^p(\mathrm{d}t, h) = f(h),$$

since $\gamma_0^p(\cdot, h)$ is the unit mass carried at $\{h\}$, cf. note to 4.13.

Next let p be even and $f \in \mathcal{M}_{p/2}^p$. Then by 12.11($b$), $H_{p/2}^p(f)$ is a carrier of ℓ_0, i.e. $H_{p/2}^p(f) = \{0\}$. Applying 12.11(c) with $k = p/2$, we clearly get

$$f_{p/2}^p(0) = 1 \cdot \int_{\mathbb{R}^p} f(t)\gamma(\mathrm{d}t, 0).$$

Thus (f). ■

An important, non-obvious, property of the marginality operator M_k^p, left out in

12.14, is that it carries $\mathcal{M}_k^p$ 'essentially onto' $\mathcal{M}(\mathcal{B}_{p-2k}, \mathcal{B}_1)$, in the sense that given $G \in \mathcal{M}(\mathcal{B}_{p-2k}, \mathcal{B}_1)$, there exists an $F \in \mathcal{M}_k^p$ such that $M_k^p(F) = G$, a.e. ℓ_{p-2k} on $\mathbb{R}^{p-2k}$. Remarkably, F can be so chosen that it works for all k. The proof of this result, 12.17 below, hinges on the lemma 12.16, to prove which we need in turn, the following rough analogue of 7.5.

12.15. Lemma. *Let $p \in \mathbb{N}_+$ & $j, k \in [0, [p/2]]$. Then with the convention 7.4,*
(*a*) $\forall \pi \in \Pi_k^p$ & $\forall h \in \mathbb{R}^{p-2k}$, $(\mathbb{R}^{2j} \times \mathbb{R}_*^{p-2j})_\pi^p(h) \subseteq \chi_{[0,j]}(k) \cdot \mathbb{R}^k$;
(*b*) $\forall h \in \mathbb{R}_*^{p-2k}$, $(\mathbb{R}^{2k} \times \mathbb{R}_*^{p-2k})_{\pi_k}^p(h) = \mathbb{R}^k$, *where π_k is the k-standard partition in Π_k^p.*

Proof. (*a*) Let $\pi \in \Pi_k^p$ and $h \in \mathbb{R}^{p-2k}$. By convention 7.4, $\mathbb{R}^k$ is an admissible version of $1 \cdot \mathbb{R}^k$. Hence for $k \in [0, j]$, taking this version, the RHS of (*a*) is $\mathbb{R}^k$ and the inclusion in (*a*) holds trivially.

Next let $k \notin [0, j]$, i.e. $j < k$. Then the RHS of (*a*) is $0 \cdot \mathbb{R}^k$, i.e. is in $\mathcal{N}_{\ell_k}$. Hence to complete the proof we have only to show that

$$\text{(I)} \qquad (\mathbb{R}^{2j} \times \mathbb{R}_*^{p-2j})_\pi^p(h) \in \mathcal{N}_{\ell_k}.$$

Proof of (I). Write $A := \mathbb{R}^{2j} \times \mathbb{R}_*^{p-2j}$, for brevity. Then since for $\pi := \{\Delta_1, \ldots, \Delta_k\}$, $\max \Delta_k \geqslant 2k + 1 > 2j + 1$, therefore $\Delta_k \in \pi \cap [2j+1, p]$. Obviously

$$I_{\Delta_k}^p := \{t : t \in \mathbb{R}^p \ \&\ t_{\min \Delta_k} = t_{\max \Delta_k}\} \subseteq \mathbb{R}^{2j} \times I_1^{p-2j}.$$

Hence, cf. 4.7 and 4.3, $\forall h \in \mathbb{R}^{p-2k}$,

$$\begin{aligned} A \cap I_\pi^p(h) \subseteq A \cap I(\pi, p) \subseteq A \cap I_{\Delta_k}^p &= (\mathbb{R}^{2j} \times \mathbb{R}_*^{p-2j}) \cap (\mathbb{R}^{2j} \times I_1^{p-2j}) \\ &\subseteq \mathbb{R}^{2j} \times (\mathbb{R}_*^{p-2j} \cap I_1^{p-2j}) = \mathbb{R}^{2j} \times \emptyset = \emptyset. \end{aligned}$$

Thus $A \cap I_\pi^p(h) = \emptyset$, and so applying $\wp_{\pi^*}$,

$$A_\pi^p(h) = \emptyset = 0 \cdot \mathbb{R}^k \in \mathcal{N}_{\ell_k}.$$

Thus (I). This establishes (*a*).

(*b*) Let $\tau = (\tau_1, \ldots, \tau_k) \in \mathbb{R}^k$ and $h \in \mathbb{R}_*^{p-2k}$. Define

$$t := (\tau_1, \tau_1, \ldots, \tau_k, \tau_k, h_1, \ldots, h_{p-2k}).$$

Then clearly $\tau = \wp_{\pi_k^*}(t)$, where $t \in \mathbb{R}^{2k} \times \mathbb{R}_*^{p-2k}$, and $t \in I_{\pi_k}^p(h)$. Thus

$$\tau \in \wp_{\pi_k^*}\{(\mathbb{R}^{2k} \times \mathbb{R}_*^{p-2k}) \cap I_{\pi_k}^p(h)\} =: (\mathbb{R}^{2k} \times \mathbb{R}_*^{p-2k})_{\pi_k}^p.$$

As this holds $\forall \tau \in \mathbb{R}^k$, we have (*b*). ∎

Combining 7.5 with the last lemma, we get the lemma required in the proof of the next theorem.

12.16. Lemma. *Let* (i) $p \in \mathbb{N}_+$, $j, k \in [0, [p/2]]$, (ii) π_j *be the j-standard partition in Π_j^p and $\pi \in \Pi_k^p$, and* (iii)

$$Z_j := I(\pi_j, p) \cap (\mathbb{R}^{2j} \times \mathbb{R}_*^{p-2j}) \subseteq \mathbb{R}^p.$$

Then with the convention 7.4,
(*a*) $\forall h \in \mathbb{R}_*^{p-2k}$, $(Z_j)_\pi^p(h) = \delta_{jk}\chi_{2^\pi}(\pi_j) \cdot \mathbb{R}^k$;
(*b*) $\forall h \in \mathbb{R}_*^{p-2k}$, $[J_{\pi,h}^p(\chi_{Z_j})](\cdot) = \delta_{jk}\chi_{2^\pi}(\pi_j)$, *a.e. ℓ_k on $\mathbb{R}^k$.*

Proof. (a) Let $h \in \mathbb{R}_*^{p-2k}$, then by (iii), the multiplicative property of the Boolean homomorphism in 4.11, and the results 7.5(a), we have

$$(1) \qquad \begin{aligned}(Z_j)_\pi^p(h) &= \{I(\pi_j,p)\}_\pi^p(h) \cap (\mathbb{R}^{2j} \times \mathbb{R}_*^{p-2j})_\pi^p(h) \\ &= \chi_{[0,k]}(j)\chi_{2^\pi}(\pi_j)\cdot\mathbb{R}^k \cap (\mathbb{R}^{2j}\times\mathbb{R}_*^{p-2j})_\pi^p(h) \\ &\subseteq \chi_{[0,k]}(j)\chi_{2^\pi}(\pi_j)\cdot\mathbb{R}^k \cap \chi_{[0,j]}(k)\cdot\mathbb{R}^k, \qquad \text{by 12.15}(a).\end{aligned}$$

Since $\chi_{[0,k]}(j)\cdot\chi_{[0,j]}(k) = \delta_{jk}$, we get

$$(2) \qquad (Z_p)_\pi^p(h) \subseteq \delta_{jk}\cdot\chi_{2^\pi}(\pi_j)\cdot\mathbb{R}^k.$$

We see that the RHS of (2) becomes ℓ_k-essentially $\emptyset$ when $j \neq k$ or $\pi_j \nsubseteq \pi$, in which case (2) becomes an equality. Thus the equality in (a) holds, if $j \neq k$ or $\pi_j \nsubseteq \pi$.

Next let $k = j$ and $\pi_j \subseteq \pi$, i.e. let $k = j$ and $\pi_k = \pi = \pi_j$. Then the RHS of (a) $= \mathbb{R}^k$. Also by 7.5(b),

$$\{I(\pi_j,p)\}_\pi^p(h) = \delta_{\pi_j,\pi}\mathbb{R}^k = 1\cdot\mathbb{R}^k$$

&

$$(\mathbb{R}^{2j}\times\mathbb{R}_*^{p-2j})_\pi^p(h) = (\mathbb{R}^{2k}\times\mathbb{R}_*^{2-pk})_{\pi_k}^p(h) = \mathbb{R}^k, \qquad \text{by 12.15}(b).$$

Thus (1) reduces to $(Z_j)_\pi^p(h) = \mathbb{R}^k \cap \mathbb{R}^k = \mathbb{R}^k$, and the equality in ($a$) again prevails. This proves (a).

(b) Recalling that, by 12.2(c) and 12.5(d), $\forall A \subseteq \mathbb{R}^p$,

$$[J_{\pi,h}^p(\chi_A)](\tau) := (\chi_A)_\pi^p(\tau,h) = \chi_{A_\pi^p(h)}(\tau), \qquad \tau\in\mathbb{R}^k,$$

we see on taking $A = Z_J$ that (b) just restates (a). ■

12.17. Main theorem. ('Lebesgue essential ontoness' of M_k^p) *Let $p \in \mathbb{N}_+$. Then $\forall k \in [0,[p/2]]$, the marginality operator M_k^p on $\mathcal{M}_k^p$ is ℓ_{p-2k} essentially onto $\mathcal{M}(\mathcal{B}_{p-2k},\mathcal{B}_1)$, and uniformly for k. More precisely, let $G_0, G_1,\dots,G_{[p/2]}$ be such that $\forall k\in[0,[p/2]]$, $G_k \in \mathcal{M}(\mathcal{B}_{p-2k},\mathcal{B}_1)$. Then*

$$\exists F \in \bigcap_{k=0}^{[p/2]} \mathcal{M}_k^p \quad \ni \quad \forall k \in [0,[p/2]], \qquad M_k^p(F) = G_k, \quad \text{a.e.} \quad \ell_{p-2k}.$$

In particular (for $k = 0$), $F = G_0$, a.e. ℓ_p.

Specifically, we can take F to be such that $\forall t\in\mathbb{R}^p$,

$$F(t) = \sum_{j=0}^{[p/2]} \rho_j\{\wp_{\pi_j^*}(t)\}G_j(t_{2j+1},\dots,t_p)\chi_{I(\pi_j,p)\cap(\mathbb{R}^{2j}\times\mathbb{R}_*^{p-2j})}(t),$$

where each $\rho_j(\cdot)$ can be any probability density on $\mathbb{R}^j$.[10]

Proof. Let $k\in[0,[p/2]]$, and grant momentarily that

$$(\text{I}) \qquad \forall h \in \mathbb{R}_*^{p-2k}, \qquad \left\{\sum_{\pi\in\Pi_k^p} J_{\pi,h}^p(F)\right\}(\cdot) = \rho_k(\cdot)G_k(h) \quad \text{on} \quad \mathbb{R}^k.$$

Then for ℓ_{p-2k} almost all $h\in\mathbb{R}^{p-2k}$, LHS(I) $= G_k(h)\rho_k(\cdot) \in L_1(\mathbb{R}^k)$. Therefore by

[10] For $j = 0$, ρ_0 is a probability density on $\mathbb{R}^0 = \{0\}$, i.e. ρ_j is the function on $\{0\}$ such that $\rho_0(0) = 1$.

12.13(d), $F \in \mathcal{M}_k^p$. By 12.14(a), $\forall h \in \mathbb{R}^{p-2k}$,

$$(1) \qquad [M_k^p(F)](h) = \chi_{H_k^p(F)}(h) \int_{\mathbb{R}^k} \left[\sum_{\pi \in \Pi_k^p} J_{\pi,h}^p(F) \right](\tau)\ell_k\,(\mathrm{d}\tau).$$

Now let $h \in \mathbb{R}_*^{p-2k} \cap H_k^p(F)$. Then by (1) and (I),

$$(2) \quad [M_k^p(F)](h) = \int_{\mathbb{R}^k} \rho_k(\tau) G_k(h) \ell_k\,(\mathrm{d}\tau) = G_k(h) \int_{\mathbb{R}^k} \rho_k(\tau)\ell_k\,(\mathrm{d}\tau) = G_k(h),$$

since $\rho_k(\cdot)$ is a probability density on $\mathbb{R}^k$. Now $\mathbb{R}_*^{p-2k}$ is a carrier of ℓ_{p-2k}, and since $F \in \mathcal{M}_k^p$, therefore by definition 12.11(b), $H_k^p(F)$ is a carrier of ℓ_{p-2k}. Thus $\mathbb{R}_*^{p-2k} \cap H_k^p(F)$ is a carrier of ℓ_{p-2k}, i.e. by (2), $M_k^p(F) = G_k$, a.e. ℓ_{p-2k}, as desired. Hence it only remains to prove (I).

Proof of (I). Define $\forall j \in [0, [p/2]]$ and $\forall t \in \mathbb{R}^p$,

$$(3) \qquad \begin{cases} g_j(t) = g_j(t_1, \ldots, t_p) := G_j(t_{2j+1}, \ldots, t_p), \\ f_j(t) := \rho_j\{\wp_{\pi_j^*}(t)\} \cdot g_j(t) \ \& \\ Z_j := I(\pi_j, p) \cap (\mathbb{R}^{2j} \times \mathbb{R}_*^{p-2j}). \end{cases}$$

Then, cf. enunciation,

$$(4) \qquad F = \sum_{j=0}^{[p/2]} f_j(t)\chi_{Z_j}(t).$$

Let $\pi \in \Pi_k^p$ and $h \in \mathbb{R}_*^{p-2k}$. Then by the linearity and multiplicativity of $J_{\pi,h}^p$,

$$\begin{aligned} J_{\pi,h}^p(F) &= \sum_{j=0}^{[p/2]} J_{\pi,h}^p(f_j) \cdot J_{\pi,h}^p(\chi_{Z_j}) = \sum_{j=0}^{[p/2]} J_{\pi,h}^p(f_j) \cdot \delta_{jk} \cdot \chi_{2^\pi}(\pi_j), \quad \text{by 12.16}(b) \\ &= J_{\pi,h}^p(f_k) \cdot \chi_{2^\pi}(\pi_k) = J_{\pi,h}^p(f_k)\delta_{\pi_k,\pi}, \end{aligned}$$

since $\chi_{2^\pi}(\pi_k) = \delta_{\pi_k,\pi}$, as π and π_k have the same cardinality. Hence

$$(5) \qquad \sum_{\pi \in \Pi_k^p} J_{\pi,h}^p(F) = \sum_{\pi \in \Pi_k^p} J_{\pi,h}^p(f_j) \cdot \delta_{\pi_k,\pi} = J_{\pi_k,h}^p(f_k).$$

Now by (3) and the multiplicative property of $J_{\pi,h}^p$, $\forall \tau \in \mathbb{R}^k$,

$$[J_{\pi_k,h}^p(f_k)](\tau) = [J_{\pi_k,h}^p(\rho_k \circ \wp_{\pi_k^*})](\tau) \cdot [J_{\pi_k,h}^p(g_k)](\tau).$$

But by 12.5(a), the first factor on the RHS is $\rho_k(\tau)$, and the second factor is by definition

$$\begin{aligned} g_k\{\theta_{\pi_k,h}^p)(\tau)\} &= g_k(\tau_1, \tau_1, \ldots, \tau_k, \tau_k, h_1, \ldots, h_{p-2k}) \\ &= G_k(h_1, \ldots, h_{p-2k}) = G_k(h), \qquad \text{by (3)}. \end{aligned}$$

Thus

$$\forall \tau \in \mathbb{R}^k, \qquad [J_{\pi_k,h}^p(f_k)](\tau) = \rho_k(\tau)G_k(h).$$

Substituting this on the RHS of (5), we get (I). ■

We next attend to symmetric f, and show that the earlier results appreciably simplify. We can sum up the situation in the following single proposition:

12.18. Proposition. *Let $p \in \mathbb{N}_+$, $k \in [0, [p/2]]$ and f on $\mathbb{R}^p$ be symmetric. Then*
(*a*) $\forall \pi \in \Pi_k^p$, $\forall \psi \in \mathrm{Perm}(p-2k)$, $\forall h \in \mathbb{R}^{p-2k}$ & $\forall \tau \in \mathbb{R}^p$, $f_\pi^p(\tau, h^\psi) = f_{\pi_k}^p(\tau, h)$;
(*b*) $\forall h \in \mathbb{R}^{p-2k}$ & $\forall \tau \in \mathbb{R}^k$,

$$\sum_{\pi \in \Pi_k^p} f_\pi^p(\tau, h) = \binom{p}{2k} \alpha_{2k} f(\tau_1, \tau_1, \ldots, \tau_k \tau_k; h);$$

(*c*) $\forall \pi \in \Pi_k^p$, $H_\pi^p(f) = H_{\pi_k}^p(f) = H_k^p(f) \in \mathcal{B}_{p-2k}^{\mathrm{sym}}$;
(*d*) $f \in \mathcal{M}_k^p$ *iff* $H_{\pi_k}^p$ *is a carrier of* ℓ_{p-2k};
(*e*) $\forall f \in \mathcal{M}_k^p$,

$$f_k^p(\cdot) = \chi_{H_{\pi_k}(f)}(\cdot) \binom{p}{2k} \alpha_{2k} \int_{\mathbb{R}^k} f_{\pi_k}^p(\tau, \cdot) \ell_k\,(\mathrm{d}\tau);$$

(*f*) f_k^p *is symmetric on* $\mathbb{R}^{p-2k}$.

Proof. (*a*) By a permutation $\phi \in \mathrm{Perm}(p)$, we can unravel the mix-up of τ_α's and h_β's in $f_\pi^p(\tau, h^\psi)$ to get $f_{\pi_k}^p(\tau, h)$. But since f is symmetric, such a permutation will not affect f. Thus (*a*).

(*b*) This follows at once from (*a*), (12.3) and (1.17).

(*c*) Since by (*a*), for each h, $f_\pi^p(\cdot, h) = f_{\pi_k}^p(\cdot, h)$, the first equality in (*c*) follows at once from the definition of $H_\pi^p(f)$ in 12.9(*d*). The second follows from this, since $H_k^p(f)$ is now the intersection of equal sets, cf. 12.12(*b*). As for its symmetry, let $\psi \in \mathrm{Perm}(p-2k)$. Then since $f_\pi^p(\cdot, h^\psi) = f_\pi^p(\cdot, h)$, we have

$$\begin{aligned} h \in \{H_\pi^p(f)\}^{\psi^{-1}} &\iff h^\psi \in H_\pi^p(f) \iff f_\pi^p(\cdot, h^\psi) \in L_1(\mathbb{R}) \\ &\iff f_\pi^p(\cdot, h) \in L_1(\mathbb{R}) \iff h \in H_\pi^p(f), \qquad \text{by } (a). \end{aligned}$$

Thus $\{H_\pi^p(f)\}^{\psi^{-1}} = H_\pi^p(f)$. Hence $H_\pi^p(f)$ is symmetric.

(*d*) This follows at once from the definition 12.11(*b*) of $\mathcal{M}_k^p$ and the last equality in (*c*).

(*e*) This follows from the equality 12.14(*a*) and the equalities in (*b*) and (*c*).

(*f*) Since f is symmetric on $\mathbb{R}^p$, therefore

$$\forall \tau \in \mathbb{R}^k, \qquad f_{\pi_k}^p(\tau, \cdot) = f(\tau_1, \tau_1, \ldots, \tau_k \tau_k, \cdot) \text{ is symmetric on } \mathbb{R}^{p-2k}.$$

Hence

$$\int_{\mathbb{R}^k} f_{\pi_k}^p(\tau, \cdot) \ell_k\,(\mathrm{d}\tau) \text{ is symmetric on } \mathbb{R}^{p-2k}.$$

But by (*c*), $\chi_{H_\pi^p}$ is symmetric on $\mathbb{R}^{p-2k}$. Hence by (*e*), f_k^p is symmetric. Thus (*f*). ∎

Finally we turn to the tensor products $f_1 \times \cdots \times f_p$ of real valued functions f_ν on $\mathbb{R}$. They are the functional analogues of the intervals $P^1 \times \cdots \times P^p$ in the set-theory in §§ 3 and 4, and play an analogous basic role. We can arrive at their sectioning by starting from 4.9(*b*) and applying the heuristic rule 12.8. The sectioning takes the following simple form, cf. (4.19):

12.19. Triviality. *Let* (i) p, k, π *and* M'_π *be as in (12.1)* & $\Delta_\alpha := \{i_\alpha, j_\alpha\}$, (ii) f, $f_1, \ldots, f_p$ *be functions on* $\mathbb{R}$ *to* $\mathbb{R}$, (iii) $\tau \in \mathbb{R}^k$ & $h \in \mathbb{R}^{p-2k}$. *Then*
(*a*)

$$\left(\bigtimes_{\nu=1}^p f_\nu\right)_\pi^p (\tau, h) = \left[\prod_{\alpha=1}^k f_{\min \Delta_\alpha}(\tau_\alpha) \cdot f_{\max \Delta_\alpha}(\tau_\alpha)\right] \cdot \prod_{\beta=1}^{p-2k} f_{m_\beta}(h_\beta);$$

(*b*) *in particular,* $(f^{\times p})^p_\pi(\tau,h) = [f^{\times k}(\tau)]^2 \cdot f^{\times(p-2k)}(h)$, *where* $f^{\times p} := f \times f \times \cdots$ *(p times).*

Proof. Obviously,

$$(i_1, j_1, \dots, i_k, j_k, m_1, \dots, m_{p-2k}) \text{ is a permutation } \phi \text{ of } (1, 2, \dots, p).$$

It follows that $\forall t = (t_1, \dots, t_p) \in \mathbb{R}^p$,

$$(t_{i_1}, t_{j_1}, \dots, t_{i_k}, t_{j_k}, t_{m_1}, \dots, t_{m_{p-2k}}) \text{ is the permutation } t^\phi \text{ of } t.$$

Now let $F := \times_{\nu=1}^p f_\nu = f_1 \times \cdots \times f_p$. Then since multiplication is commutative, it follows that $\forall t = (t_1, \dots, t_p) \in \mathbb{R}^p$,

$$(1) \qquad F(t) := \prod_{\nu=1}^{p} f_\nu(t_\nu) = \left[\prod_{\alpha=1}^{k} f_{i_\alpha}(t_{i_\alpha}) f_{j_\alpha}(t_{j_\alpha})\right] \cdot \prod_{\beta=1}^{m-2k} f_{m_\beta}(t_{m_\beta}).$$

Now let $\tau = (\tau, \dots, \tau_k) \in \mathbb{R}^k$ & $h = (h_1, \dots, h_{p-2k}) \in \mathbb{R}^{p-2k}$. Then by definition 12.2,

$$(2) \qquad F^p_\pi(\tau, h) := F(t), \quad \text{where} \quad t = \theta^p_{\pi,h}(\tau),$$

i.e. where t is given by

$$(3) \qquad \forall \alpha \in [1,k], \quad t_{i_\alpha} = \tau_\alpha = \tau_{j_\alpha} \quad \& \quad \forall \beta \in [1, p-2k], \quad t_{m_\beta} = h_\beta.$$

Substituting from (3) into (1), and appealing to (2), we get the equality in (*a*).

(*b*) This follows from (*a*) on setting $f_1 = \cdots = f_p = f$. ■

We next address the question as to when the tensor product $f_1 \times \cdots \times f_p$ of functions $f_1, \dots, f_p$ on $\mathbb{R}$ to $\mathbb{R}$ is in $\mathcal{M}_{p,k}$, cf. definition 12.11(*b*). We have the following elementary lemma, where (*b*) and (*c*) are the analogues of (4.20) in which linear combinations of tensor products replace the simple functions (i.e. replace linear combinations of indicators):

12.20. Lemma. *Let* (i) $p \in \mathbb{N}_+$ & $k \in [0, [p/2]]$, (ii) $f_1, \dots, f_p \in L_2(\mathbb{R})$. *Then*

(*a*)

$$\bigtimes_{\nu=1}^{p} f_\nu \in \mathcal{M}^p_k;$$

(*b*) $\forall \pi \in \Pi^p_k$ & $\forall h \in \mathbb{R}^{p-2k}$,

$$\int_{\mathbb{R}^k} \left(\bigtimes_{\nu=1}^{p} f_\nu\right)^p_\pi (\tau, h) \ell_k(\mathrm{d}\tau) = \prod_{\Delta \in \pi} (f_{\min \Delta}, f_{\max \Delta})_{L_2(\mathbb{R})} \cdot \left(\bigtimes_{m \in M'_\pi} f_m\right)(h);$$

(*c*) $\forall h \in \mathbb{R}^{p-2k}$,

$$\left(\bigtimes_{\nu=1}^{p} f_\nu\right)^p_k (h) = \sum_{\pi \in \Pi^p_k} \left\{\prod_{\Delta \in \pi} (f_{\min \Delta}, f_{\max \Delta})_{L_2(\mathbb{R})} \cdot \left(\bigtimes_{m \in M'_\pi} f_m\right)(h)\right\}.$$

(*d*) *in particular, for even p,*

$$\left(\bigtimes_{\nu=1}^{p} f_\nu\right)^p_{p/2} (0) = \sum_{\pi \in \Pi_{[1,p]}} \left\{\prod_{\Delta \in \pi} (f_{\min \Delta}, f_{\max \Delta})_{L_2(\mathbb{R})}\right\}.$$

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Proof. (*a*) Since $F := \times_{\nu=1}^{p} f_\nu \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$, therefore by definition 12.11(*b*), we need only show that $H_k^p(F)$ is a carrier of ℓ_{p-2k}. We shall show in fact that $H_k^p(F) = \mathbb{R}^{p-2k}$, i.e.

$$\text{(I)} \qquad \forall \pi \in \Pi_k^p \quad \& \quad \forall h \in \mathbb{R}^{p-2k}, \qquad \left(\bigtimes_{\nu=1}^{p} f_\nu \right)_\pi^p (\cdot, h) \in L_1(\mathbb{R}^k).$$

Proof of (I). Let π and M'_π be as in 12.1. Then by 12.19(*a*), $\forall h \in \mathbb{R}^{p-2k}$,

$$\begin{aligned}
&\int_{\mathbb{R}^k} \left| \left(\bigtimes_{\nu=1}^{p} f_\nu \right)_\pi^p (\tau, h) \right| \ell_k\,(\mathrm{d}\tau) \\
&\quad = \int_{\mathbb{R}^k} \left| \prod_{\alpha=1}^{k} f_{\min \Delta_\alpha}(\tau_\alpha) \cdot f_{\max \Delta_\alpha}(\tau_\alpha) \right| \ell_k\,(\mathrm{d}\tau) \cdot \prod_{\beta=1}^{p-2k} |f_{m_\beta}(h_\beta)| \\
&\quad = \prod_{\alpha=1}^{k} \int_{\mathbb{R}^k} |f_{\min \Delta_\alpha}(\tau_\alpha) \cdot f_{\max \Delta_\alpha}(\tau_\alpha)| \ell_1\,(\mathrm{d}\tau_\alpha) \cdot \prod_{\beta=1}^{p-2k} |f_{m_\beta}(h_\beta)| \\
&\quad \leqslant \prod_{\alpha=1}^{k} |f_{\min \Delta_\alpha}|_{2,\ell_1} |f_{\max \Delta_\alpha}|_{2,\ell_1} \cdot \prod_{\beta=1}^{p-2k} |f_{m_\beta}(h_\beta)| < \infty,
\end{aligned}$$

by the Schwartz inequality and (ii). Thus (I) holds, and (*a*) is proved.

(*b*) Let $\pi \in \Pi_k^p$ & $h \in \mathbb{R}^{p-2k}$. Then by 12.19(*a*),

$$\begin{aligned}
&\int_{\mathbb{R}^k} \left(\bigtimes_{\nu=1}^{p} f_\nu \right)_\pi^p (\tau, h) \ell_k\,(\mathrm{d}\tau) \\
&\quad = \int_{\mathbb{R}^k} \left[\prod_{\alpha=1}^{k} f_{\min \Delta_\alpha}(\tau_\alpha) \cdot f_{\max \Delta_\alpha}(\tau_\alpha) \right] \cdot \prod_{\beta=1}^{p-2k} f_{m_\beta}(h_\beta) \cdot \ell_k\,(\mathrm{d}\tau) \\
&\quad = \prod_{\alpha=1}^{k} \int_{\mathbb{R}^k} f_{\min \Delta_\alpha}(\tau_\alpha) \cdot f_{\max \Delta_\alpha}(\tau_\alpha) \ell_1\,(\mathrm{d}\tau_\alpha) \cdot \prod_{\beta=1}^{p-2k} f_{m_\beta}(h_\beta) \\
&\quad = \prod_{\alpha=1}^{k} (f_{\min \Delta_\alpha}, f_{\max \Delta_\alpha})_{L_2(\mathbb{R})} \cdot \left(\bigtimes_{\beta=1}^{p-2k} f_{m_\beta} \right)(h) = \mathrm{RHS}(b).
\end{aligned}$$

Thus (*b*).

(*c*) This follows from (*b*) and by 12.10(*c*). The result (*d*) is an obvious special case. ■

An important special case of the last result, obtained on setting $f_1 = \cdots = f_p = f$, is the following:

12.21. Corollary. *Let* $p \in \mathbb{N}_+$, $k \in [0, [p/2]]$ & $f \in L_2(\mathbb{R})$. *Then*

(*a*) $f^{\times p} \in \mathcal{M}_k^p$;

(*b*) $\forall \pi \in \Pi_k^p$ & $\forall h \in \mathbb{R}^{p-2k}$,

$$\int_{\mathbb{R}^k} (f^{\times p})_\pi^p (\tau, h) \ell_k\,(\mathrm{d}\tau) = |f|_{2,\ell_1}^{2k} \cdot f^{\times(p-2k)}(h);$$

(c) $\forall h \in \mathbb{R}^{p-2k}$, $(f^{\times p})^p_k(h) = \binom{p}{2k}\alpha_{2k}|f|^{2k}_{2,\ell_1} \cdot f^{\times(p-2k)}(h)$.

13. Integrability and integration with respect to the measure ξ_p

The greater complexity of the covariance structure of the measures ξ_p in relation to the measures η_p is reflected in a greater complexity of the classes $\mathcal{P}_{1,\xi_p}$ and the operators $\mathbb{E}_{\xi_p}$, vis-à-vis $\mathcal{P}_{1,\eta_p}$ and $\mathbb{E}_{\eta_p}$ discussed in § 10. We abide by the definitions and results in (A.9)–(A.26), where now ρ is to be ξ_p.

For the relationship between the spaces $\mathcal{P}_{1,\xi_p}$ and $\mathcal{P}_{1,\eta_p}$ note that from the inequalities

(13.1) $$\forall f \in \mathcal{M}(\mathcal{B}_p, B\ell(\mathbb{R})), \qquad |f|_{1,\eta_p} \leqslant |f|_{1,\xi_p} \quad \& \quad |f|_{1,\zeta_p} \leqslant |f|_{1,\xi_p},$$

which by (A.9) are simple consequences of 9.13(h), it follows at once that

(13.2) $$\mathcal{P}_{1,\xi_p} \subseteq \mathcal{P}_{1,\eta_p} = L_2(\mathbb{R}^p) \quad (\text{cf. } 10.5) \qquad \& \quad \mathcal{P}_{1,\xi_p} \subseteq \mathcal{P}_{1,\zeta_p}.$$

But the inclusions in (13.2) are proper for $p \geqslant 2$, as the following example shows:

13.3. *Example.* Let $p \in \mathbb{N}_+$ be even. Then $\chi_{I^p_{p/2}} \in L_2(\mathbb{R}^p)\backslash\mathcal{P}_{1,\xi_p}$.

Proof. Since $I^p_{p/2} \subseteq I^p_1 \in \mathcal{N}_{\ell_p}$, cf. 7.1($a$) and (4.14), so obviously $f := \chi_{I^p_{p/2}} \in L_2(\mathbb{R}^p)$. Next, let $\forall n \in \mathbb{N}_+$, D_n be the part of $I^p_{p/2}$ inside the box $[0,n]^p$, i.e. $D_n := [0,n] \cap I^p_{p/2}$. Then $D_n \in \mathcal{D}_p$ and cf. (A.13)

(1) $$|f|_{1,\xi_p} = s_{\xi_p}(I^p_{p/2}) \geqslant s_{\xi_p}(D_n) \geqslant |\xi_p(D_n)|_{\mathcal{L}_2}.$$

But by (8.7) and (8.8),

$$\xi_p(D_n) = \mathbb{E}_{\mathbb{P}}\{\xi_p([0,n]^p)\} \cdot 1(\cdot) = \alpha_{p/2} n^{p/2} \cdot 1(\cdot).$$

Hence $|\xi_p(D_n)| = \alpha_{p/2} n^{p/2}$. Thus by (1), $|f|_{1,\xi_p} = \infty$, i.e. $f \notin \mathcal{P}_{1,\xi_p}$. ■

The specific part of $L_2(\mathbb{R}^p)$ that constitutes $\mathcal{P}_{1,\xi_p}$, emerges from the next lemma, in which $\xi_{p,k}(\cdot) := \text{proj}(\xi_p(\cdot)|\mathcal{S}_{\eta_{p-2k}})$ on $\mathcal{D}_p$, cf. (9.15):

13.4. Lemma. *Let $p \in \mathbb{N}_+$ & $k \in [0,[p/2]]$. Then $\forall f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$,*

$$\frac{1}{[p/2]} \sum_{k=0}^{[p/2]-1} |f|_{1,\xi_{p,k}} \leqslant |f|_{1,\xi_p} \leqslant \sum_{k=0}^{[p/2]} |f|_{1,\xi_{p,k}}.$$

Proof. We first assert that $\forall x' \in (\mathcal{L}_2)'$ & $\forall A \in \mathcal{B}_p$,

(I) $$\frac{1}{[p/2]} \sum_{k=0}^{[p/2]-1} |x' \circ \xi_{p,k}|(A) \leqslant |x' \circ \xi_p|(A) \leqslant \sum_{k=0}^{[p/2]} |x' \circ \xi_{p,k}|(A).$$

Proof of (I). Let $x' \in (\mathcal{L}_2)'$ & $A \in \mathcal{B}_p$. Then, since by (9.10), $\xi_p = \sum_{k=0}^{[p/2]} \xi_{p,k}$, therefore

$$\forall D \in \mathcal{D}_p, \qquad x' \circ \xi_p(D) = \sum_{k=0}^{[p/2]} x' \circ \xi_{p,k}(D),$$

whence follows the second inequality in (I), namely,

$$(1) \qquad |x' \circ \xi_p|(A) \leqslant \sum_{k=0}^{[p/2]} |x' \circ \xi_{p,k}|(A).$$

Next from 9.16(b), for $0 \leqslant k < [p/2]$,

$$(2) \qquad |x' \circ \xi_{p,k}|(A) = |x' \circ \xi_p|\{A \cap (I_k^p \backslash I_{k+1}^p)\} \leqslant |x' \circ \xi_p|(A).$$

Thus

$$\sum_{k=0}^{[p/2]-1} |x' \circ \xi_{p,k}|(A) \leqslant [p/2] \cdot |x' \circ \xi_p|(A),$$

and division by $[p/2]$ yields the first inequality in (I). Thus (I).

Now let $f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$. Then by (2), $\forall k \in [0, [p/2]-1]$,

$$\begin{aligned} \int_{\mathbb{R}^p} |f(t)| \cdot |x' \circ \xi_{p,k}| \, (\mathrm{d}t) &\leqslant \int_{\mathbb{R}^p} |f(t)| \cdot |x' \circ \xi_p| \, (\mathrm{d}t) \\ &\leqslant \sum_{j=0}^{[p/2]} \int_{\mathbb{R}^p} |f(t)| \cdot |x' \circ \xi_{p,j}| \, (\mathrm{d}t), \quad \text{by (1)}. \end{aligned}$$

Taking the sup for $|x'| \leqslant 1$ in all three terms, we get

$$\forall k \in [0, [p/2]-1], \qquad |f|_{1,\xi_{p,k}} \leqslant |f|_{1,\xi_p} \leqslant \sum_{k=0}^{[p/2]} |f|_{1,\xi_{p,k}}.$$

Summing over k in the first of these inequalities and dividing by $[p/2]$, we get

$$\frac{1}{[p/2]} \sum_{k=0}^{[p/2]-1} |f|_{1,\xi_{p,k}} \leqslant |f|_{1,\xi_p} < \sum_{k=0}^{[p/2]} |f|_{1,\xi_{p,k}}.$$

■

It follows at once from the last lemma, and from (9.10) and the linearity of $\mathbb{E}_\rho(f)$ as an operator acting on the measure ρ, the inclusion $\operatorname{Range} \mathbb{E}_{\xi_{p,k}} \subseteq \mathcal{S}_{\xi_{p,k}}$, and the orthogonality of the $\mathcal{S}_{\xi_{p,k}}$ that

$$(13.5) \quad \begin{cases} \forall p \in \mathbb{N}_+, \quad \mathcal{P}_{1,\xi_p} = \bigcap_{k=0}^{[p/2]} \mathcal{P}_{1,\xi_{p,k}} \\ \& \\ \forall f \in \mathcal{P}_{1,\xi_p}, \quad \mathbb{E}_{\xi_p}(f) = \sum_{k=0}^{[p/2]} \mathbb{E}_{\xi_{p,k}}(f), \quad \mathbb{E}_{\xi_{p,j}}(f) \perp \mathbb{E}_{\xi_{p,k}}(f), \quad j \neq k. \end{cases}$$

These equalities reveal the centrality of the measure $\xi_{p,k}$, and bring up the question as to what are the classes $\mathcal{P}_{1,\xi_{p,k}}$ and the operators $\mathbb{E}_{\xi_{p,k}}$—questions, which are interesting by virtue of the formula

$$\forall D \in \mathcal{D}_p, \qquad \xi_{p,k}(D) = \int_{\mathbb{R}^{p-2k}} \gamma_k^p(D, h) \eta_{p-2k} \, (\mathrm{d}h),$$

established in 11.7. These questions are answered in theorem 13.10 and corollary 13.12, on the basis of the results in Appendix B.

We first note that since by 6.1 and 9.16(c), ξ_p and $\xi_{p,k}$ are invariant under the group on $\mathcal{B}_p$ induced by Perm(p), we infer from lemma A.35 that:

$$(13.6)\quad \begin{cases} \forall p \in \mathbb{N}_+ \quad \& \quad \forall k \in [0,[p/2]], \\ f \in \mathcal{P}_{1,\xi_p} \Longrightarrow \forall \phi \in \text{Perm}(p), \quad f^\phi \in \mathcal{P}_{1,\xi_p} \quad \& \quad \mathbb{E}_{\xi_p}(f^\phi) = \mathbb{E}_{\xi_p}(f), \\ f \in \mathcal{P}_{1,\xi_{p,k}} \Longrightarrow \forall \phi \in \text{Perm}(p), \quad f^\phi \in \mathcal{P}_{1,\xi_{p,k}} \quad \& \quad \mathbb{E}_{\xi_{p,k}}(f^\phi) = \mathbb{E}_{\xi_{p,k}}(f). \end{cases}$$

Moreover ($\tilde{}$ denoting symmetrization), (13.6) entails that

$$(13.7)\quad \begin{cases} f \in \mathcal{P}_{1,\xi_p} \Longrightarrow \tilde{f} \in \mathcal{P}_{1,\xi_p} \quad \& \quad \mathbb{E}_{\xi_p}(\tilde{f}) = \mathbb{E}_{\xi_p}(f), \\ f \in \mathcal{P}_{1,\xi_{p,k}} \Longrightarrow \tilde{f} \in \mathcal{P}_{1,\xi_{p,k}} \quad \& \quad \mathbb{E}_{\xi_{p,k}}(\tilde{f}) = \mathbb{E}_{\xi_{p,k}}(f). \end{cases}$$

The example cited in the note to 10.3 shows that $\tilde{f} \in \mathcal{P}_{1,\xi_p} \not\Longrightarrow f \in \mathcal{P}_{1,\xi_p}$. Consequently in studying $\mathcal{P}_{1,\xi_{p,k}}$ and $\mathbb{E}_{\xi_{p,k}}$, we may take f to be symmetric only with caution.

We must now note that by B.9, the measure $\sigma := \eta_{p-2k}$ satisfies the restraint (B.2) on letting $\mathcal{H} = \mathcal{L}_2$ and $q = p - 2k$, and that the canonical coefficients $\gamma_k^p(\cdot,\cdot)$ have all the attributes of the Markovian type kernels listed in (B.3). More fully,

13.8. Lemma. *Let $p \in \mathbb{N}_+$ & $k \in [0,[p/2]]$. Then $K(\cdot,\cdot) := \gamma_k^p(\cdot,\cdot)$ satisfies (B.3), i.e.*

(a) *$\gamma_k^p(\cdot,\cdot)$ is a function on $\mathcal{D}_p \times \mathbb{R}^{p-2k}$ to $\mathbb{R}_{0+}$;*
(b) *$\forall h \in \mathbb{R}^{p-2k}$, $\gamma_k^p(\cdot,h) \in \text{CA}(\mathcal{D}_p,\mathbb{R}_{0+})$;*
(c) *$\forall D \in \mathcal{D}_p$, $\gamma_k^p(D,\cdot) \in \mathcal{P}_{1,\eta_{p-2k}}$;*
(d) *$\forall D \in \mathcal{D}_p^{\text{sym}}$, $\gamma_k^p(D,\cdot)$ is symmetric on $\mathbb{R}^{p-2k}$.*

Proof. (a)–(c) are clear from 4.16(a), (b); and (d) restates 6.10(b). ■

Furthermore, by theorem 11.7, the measure $\rho(\cdot)$, determined by the kernel $\gamma_k^p(\cdot,\cdot)$ is $\xi_{p,k}$, which by 9.16(c) is permutation invariant, i.e. we have

$$(13.9)\quad \begin{cases} \forall k \in [0,[p/2]] \quad \& \quad \forall D \in \mathcal{D}_p, \quad \xi_{p,k}(D) = \displaystyle\int_{\mathbb{R}^{p-2k}} \gamma_k^p(D,h)\eta_{p-2k}(\mathrm{d}h) \\ \& \quad \forall \phi \in \text{Perm}(p), \quad \xi_{p,k}(D^\phi) = \xi_{p,k}(D). \end{cases}$$

Thus all the premises imposed on the kernel $K(\cdot,\cdot)$ and the measure ρ in main theorem B.8 are satisfied (q now being $p - 2k$), and from this theorem we conclude (recalling the definition 12.11 of $\mathcal{M}_k^p$ and f_k^p):

13.10. Main theorem. (on $\mathcal{P}_{1,\xi_{p,k}}$ & $\mathbb{E}_{\xi_{p,k}}$) *Let $p \in \mathbb{N}_+$ & $k \in [0,[p/2]]$. Then*

(a) $\mathcal{P}_{1,\xi_{p,k}} = \{f : f \in \mathcal{M}_k^p \;\&\; |f|_k^p(\cdot) \in \mathcal{P}_{1,\eta_{p-2k}}\}$;
(b) $\forall f \in \mathcal{P}_{1,\xi_{p,k}}$,

$$\mathbb{E}_{\xi_{p,k}}(f) = \int_{\mathbb{R}^{p-2k}} \left\{ \int_{\mathbb{R}^p} f(t)\gamma_k^p(\mathrm{d}t,h) \right\} \eta_{p-2k}(\mathrm{d}h) = \mathbb{E}_{\eta_{p-2k}}(f_k^p);$$

(c) $\forall f \in \mathcal{P}_{1,\xi_{p,k}}^{\text{sym}}$, $\exists \tilde{N} \in \mathcal{N}_{p-2k}^{\text{sym}}$ *such that $f_k^p(\cdot)$ is symmetric on $\mathbb{R}^{p-2k} \backslash N$.*

Theorem 12.10(c) allows us to express the condition for $\xi_{p,k}$ integrability as well as the $\xi_{p,k}$ integral, in terms of the Lebesgue measure ℓ_k, in the following more usable forms:

13.11. Theorem. *Let $p \in \mathbb{N}_+$ & $k \in [0, [p/2]]$. Then*
(*a*) $f \in \mathcal{P}_{1,\xi_{p,k}}$ *iff* $f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$ & $\forall \pi \in \Pi_k^p$,

$$\int_{\mathbb{R}^{p-2k}} \left[\int_{\mathbb{R}^k} |f_\pi^p(\tau, h)| \ell_k\,(\mathrm{d}\tau) \right]^2 \ell_{p-2k}\,(\mathrm{d}h) < \infty.$$

(*b*) $\forall f \in \mathcal{P}_{1,\xi_{p,k}}$,

$$\mathbb{E}_{\xi_{p,k}}(f) = \int_{\mathbb{R}^{p-2k}} \left[\int_{\mathbb{R}^k} \left\{ \sum_{\pi \in \Pi_k^p} f_\pi^p(t, h) \right\} \ell_k\,(\mathrm{d}t) \right] \eta_{p-2k}\,(\mathrm{d}h);$$

(*c*) $\forall f \in \mathcal{P}^{\mathrm{sym}}_{1,\xi_{p,k}}$,

$$\mathbb{E}_{\xi_{p,k}}(f) = \binom{p}{2k} \alpha_{2k} \int_{\mathbb{R}^{p-2k}} \left\{ \int_{\mathbb{R}^k} f_{\pi_k}^p(\tau, h) \ell_k\,(\mathrm{d}t) \right\} \eta_{p-2k}\,(\mathrm{d}h).$$

Proof. (*a*) By 12.14(*a*) and the last equality in 12.5(*a*),

$$|f|_k^p(\cdot) = \sum_{\pi \in \Pi_k^p} \int_{\mathbb{R}^k} |f_\pi^p(\tau, \cdot)| \ell_k\,(\mathrm{d}\tau), \quad \text{a.e.} \quad \ell_{p-2k}, \quad \text{on} \quad \mathbb{R}^{p-2k}.$$

It follows readily that $|f|_k^p \in L_2(\mathbb{R}^{p-2k})$ iff each integral on the RHS is in $L_2(\mathbb{R}^{p-2k})$, i.e. iff the condition in (*a*) holds. Since $L_2(\mathbb{R}^{p-2k}) = \mathcal{P}_{1,\eta_{p-2k}}$, cf. 10.5(*a*) we have (*a*).

(*b*) On the RHS of 13.10(*b*), replace $\int_{\mathbb{R}^k} f(t) \gamma_k^p\,(\mathrm{d}t, h)$ by the expression in 12.10(*c*). This yields (*b*).

(*c*) We repeat the steps in (*b*) except for using 12.18(*e*) instead of 12.10(*c*). ■

Finally, reverting to (13.5), recalling as noted after (9.15), that $\mathcal{P}_{1,\xi_{p,0}} = \mathcal{P}_{1,\eta_p} = L_2(\mathbb{R}^p)$, and appealing to theorems 13.10 and 13.11, we can state the condition for ξ_p-integrability and the value of the ξ_p-integral as follows:

13.12. Main theorem. (on $\mathcal{P}_{1,\xi_p}$ & $\mathbb{E}_{\xi_p}$) *Let $p \in \mathbb{N}_+$. Then $f \in \mathcal{P}_{1,\xi_p}$ iff*

$$f \in \bigcap_{k=1}^{[p/2]} \mathcal{M}_k^p \quad \& \quad \forall k \in [0, [p/2]], \quad |f|_k^p \in \mathcal{P}_{1,\eta_{p-2k}},$$

i.e. $\forall k \in [0, [p/2]]$, $f \in \mathcal{M}_{p,k}$ & $\forall \pi \in \Pi_k^p$,

$$\int_{\mathbb{R}^{p-2k}} \left\{ \int_{\mathbb{R}^k} |f_\pi^p(\tau, h)| \ell_k\,(\mathrm{d}\tau) \right\}^2 \ell_{p-2k}\,(\mathrm{d}h) < \infty.$$

Moreover, we have the orthogonal expansion:

$$\forall f \in \mathcal{P}_{1,\xi_p}, \qquad \mathbb{E}_{\xi_p}(f) = \sum_{k=0}^{[p/2]} \mathbb{E}_{\eta_{p-2k}}(f_k^p).$$

Note. As the reader can easily check, the condition in 13.12 corresponding to $k = 0$ is just

$$\int_{\mathbb{R}^p} |f(h)| \ell_p\,(\mathrm{d}h) < \infty, \quad \text{i.e.} \quad f \in L_2(\mathbb{R}^p);$$

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and for $k = p/2$ when p is even, it is that

$$\forall \pi \in \Pi_{[1,p]}, \qquad \int_{\mathbb{R}^{p/2}} |f\{\theta^p_{\pi,0}(\tau)\}| \ell_{p/2}\,(\mathrm{d}\tau) < \infty.$$

For symmetric f, the condition for membership in $\mathcal{P}_{1,\xi_p}$ simplifies appreciably. We have

13.13. Corollary. (Symmetric f) *Let $p \in \mathbb{N}_+$ and f on $\mathbb{R}^p$ be symmetric. Then*

(*a*) *$f \in \mathcal{P}_{1,\xi_p}$ iff $f \in \bigcap_{k=1}^{[p/2]} \mathcal{M}^p_k$, and $\forall k \in [0,[p/2]]$,*

$$\int_{\mathbb{R}^{p-2k}} \left\{ \int_{\mathbb{R}^k} |f(\tau_1, \tau_1, \ldots, \tau_k \tau_k; h)| \ell_k\,(\mathrm{d}\tau) \right\}^2 \ell_{p-2k}\,(\mathrm{d}h) < \infty;$$

(*b*) $\forall f \in \mathcal{P}^{\mathrm{sym}}_{1,\xi_p}$

$$\mathbb{E}_{\xi_p}(f) = \sum_{k=0}^{[p/2]} \binom{p}{2k} \alpha_{2k} \int_{\mathbb{R}^{p-2k}} \left\{ \int_{\mathbb{R}^k} f^p_{\pi_k}(\tau, h) \ell_k\,(\mathrm{d}\tau) \right\} \eta_{p-2k}\,(\mathrm{d}h).$$

Proof. Part (*a*) is clear from 13.12, 12.18(*a*) and (12.3). Part (*b*) is clear from 13.12 and 12.18(*e*). ■

The covariances of the integrals $\mathbb{E}_{\xi_p}(f)$ are readily had from the equality in 13.12:

13.14. Proposition. (Covariances of the $\mathbb{E}_{\xi_p}(f)$) *Let* (i) $p, q \in \mathbb{N}_{0+}$, (ii) $f \in \mathcal{P}_{1,\xi_p}$, $g \in \mathcal{P}_{1,\xi_q}$. *Then*

(*a*) *if $q < p$ and $p - q$ is even, we have*

$$(\mathbb{E}_{\xi_p}(f), \mathbb{E}_{\xi_q}(g)) = \sum_{k=0}^{[q/2]} (q-2k)! \int_{\mathbb{R}^{p-2k}} \tilde{f}^p_{\frac{1}{2}(p-q)+k}(h) \tilde{g}^p_k(h) \ell_{q-2k}\,(\mathrm{d}h);$$

(*b*) *if $p + q$ is odd, we have $\mathbb{E}_{\xi_p}(f) \perp \mathbb{E}_{\xi_q}(g)$;*

(*c*)

$$|\mathbb{E}_{\xi_p}(f)|^2 = \sum_{k=0}^{[p/2]} (p-2k)! \int_{\mathbb{R}^{p-2k}} |\tilde{f}^p_k(h)|^2 \ell_{p-2k}\,(\mathrm{d}h);$$

(*d*) *Null space $(\mathbb{E}_{\xi_p}) = \{f : f \in \mathcal{P}_{1,\xi_\pi}\ \&\ \forall k \in [0, p/2],\ \tilde{f}^p_k(\cdot) = 0,\ \text{a.e. } \ell_{p-2k}\}$.*

Proof. (*a*) The proof is routine. By appealing to the first result in (13.7), we can deal with $(\mathbb{E}_{\xi_p}(\tilde{f}), \mathbb{E}_{\eta_q}(\tilde{f}))$. We compute this, using the last equality 13.12. We then note that by 12.13(*a*), $\tilde{f}^p_j$, $\tilde{f}^q_k$ are symmetric and we simplify further by means of 10.3(*a*).

(*b*) is obvious since by (A.29) and 8.6, $\operatorname{Range} \mathbb{E}_{\xi_p} \subseteq \mathcal{S}_{\xi_p} \perp \mathcal{S}_{\xi_q} \supseteq \operatorname{Range} \mathbb{E}_{\xi_q}$.

(*c*) We take $q = p$ and $g = f$ in (*a*).

(*d*) This follows at once from (*c*). ■

The equality in 13.14(*a*) corresponds to the covariance equality for sets given in 5.3. For on letting $f = \chi_D$ and $g = \chi_E$ in 12.18, we get the equality in 5.3 on noting the formula in 5.2(*c*).

For the expectation of the integrals $\mathbb{E}_{\xi_p}(f)$, we have:

13.15. Proposition. *Let $p \in \mathbb{N}_{0+}\ \&\ f \in \mathcal{P}_{1,\xi_p}$. Then*

(*a*) *if* p *is odd,* $\mathbb{E}_{\mathbb{P}}\{\mathbb{E}_{\xi_p}(f)\} = 0$*;*
(*b*) *if* p *is even,* $\mathbb{E}_{\mathbb{P}}\{\mathbb{E}_{\xi_p}(f)\} = f^p_{p/2}(0)$.

Proof. (*a*) Let $p \in \mathbb{N}_{0+}$. Then by (A.33),

(1) $$\mathbb{E}_{\mathbb{P}}\{\mathbb{E}_{\xi_p}(f)\} = \int_{\mathbb{R}^p} f(t)\cdot(\mathbb{E}_{\mathbb{P}}\circ\xi_p)\,(\mathrm{d}t).$$

If p is odd, then by 5.9(*b*), $\mathbb{E}_{\mathbb{P}}\circ\xi_p = 0$ on $\mathcal{D}_p$, and we get (*a*).

(*b*) If p is even, then by 5.9(*a*), $(\mathbb{E}_{\mathbb{P}}\circ\xi_p)(D) = \gamma^p_{p/2}(D,0)$, and therefore

$$\mathrm{RHS}(1) = \int_{\mathbb{R}^p} f(t)\gamma^p_{p/2}(\mathrm{d}t,0) = f^p_{p/2}(0), \qquad \text{by } 12.14(f).$$

■

The integral analogue of the projection theorem 11.7 reads as follows:

13.16. Proposition. *Let* $p \in \mathbb{N}_+$ *&* $k \in [0,[p/2]]$*. Then*
(*a*) $\forall f \in \mathcal{P}_{1,\xi_p}$, $\mathrm{Proj}(\mathbb{E}_{\xi_p}(f)|\mathcal{S}_{\eta_{p-2k}}) = \mathbb{E}_{\xi_{p,k}}(f) = \mathbb{E}_{\eta_{p-2k}}(f^p_k)$;
(*b*) $\forall f \in \mathcal{P}_{1,\xi_p}$, $\mathrm{Proj}(\mathbb{E}_{\xi_p}(f)|\mathcal{S}_{\eta_p}) = \mathbb{E}_{\eta_p}(f)$.

Proof. (*a*) Let T be the projection operator on $\mathcal{L}_2$ onto $\mathcal{S}_{\eta_{p-2k}}$. Then $\forall D \in \mathcal{D}_p$, $T\{\xi_p(D)\} = \xi_{pk}(D)$, by (9.15), i.e. $T\circ\xi_p = \xi_{pk}$. Hence for $f \in \mathcal{P}_{1,\xi_p}$, by (A.32),

$$\mathrm{LHS}(a) = T\{\mathbb{E}_{\xi_p}(f)\} = \mathbb{E}_{T\circ\xi_p}(f) = \mathbb{E}_{\xi_{pk}}(f) = \mathbb{E}_{\eta_{p-2k}}(f^p_k) \qquad \text{by } 13.10(b).$$

Thus (*a*).

(*b*) This follows on taking $k = 0$ and on noting that $f^p_0 = f$, cf. 12.14(*f*). ■

Our next goal in this section is to show that the range of the integral operator $\mathbb{E}_{\xi_p}$ is closed in $\mathcal{L}_2$, and therefore by (A.29) equals the closed subspace spanned by the values of the measure ξ_p itself, and to draw from it a result on liftings. In full analogy with the equality (10.6) for η_p, we have:

13.17. Theorem. $\forall p \in \mathbb{N}_+$, $\mathrm{Range}\,\mathbb{E}_{\xi_p} = \mathcal{S}_{\xi_p}$ *is closed.*

Proof. Obviously $\mathrm{Range}\,\mathbb{E}_{\xi_p} \subseteq \mathcal{S}_{\xi_p}$. Hence we have only to show the reverse inclusion, i.e. that

(I) $$\forall x \in \mathcal{S}_{\xi_p}, \qquad \exists F \in \mathcal{P}_{1,\xi_p} \ni \mathbb{E}_{\xi_p}(F) = x.$$

Proof of (I). Let $x \in \mathcal{S}_{\xi_p}$. Then since by (9.9),

$$\mathcal{S}_{\xi_p} = \sum_{k=0}^{[p/2]} \mathcal{S}_{\eta_{p-2k}}, \qquad \mathcal{S}_{\eta_i} \perp \mathcal{S}_{\eta_j}, \qquad i \neq j,$$

therefore

(1) $$\forall k \in [0,[p/2]], \quad \exists_1 x_k \in \mathcal{S}_{\eta_{p-2k}} \ni \sum_{k=0}^{[p/2]} x_k = x.$$

By (10.6),

(2) $$\forall k \in [0,[p/2]], \quad \exists_1 G_k \in L_2(\mathbb{R}^{p-2k}) \ni \mathbb{E}_{\eta_{p-2k}}(G_k) = x_k.$$

Now by the main theorem 12.17 we know that

$$\exists F \in \bigcap_{k=1}^{[p/2]} \mathcal{M}^p_k \quad \ni \quad \forall k \in [0,[p/2]], \qquad F^p_k = G_k, \quad \text{a.e.} \quad \ell_{p-2k} \quad \text{on} \quad \mathbb{R}^{p-2k}.$$

Since $G_k \in L_2(\mathbb{R}^{p-2k})$, each $F_k^p \in L_2(\mathbb{R}^{p-2k})$. Thus by theorem 13.12, $F \in \mathcal{P}_{1,\xi_p}$. Furthermore, by theorem 13.12,

$$\mathbb{E}_{\xi_p}(F) = \sum_{k=0}^{[p/2]} \mathbb{E}_{\eta_{p-2k}}(F_k^p) = \sum_{k=0}^{[p/2]} \mathbb{E}_{\eta_{p-2k}}(G_k) = x, \quad \text{by (1) and (2).}$$

■

The last theorem allows us to settle the lifting problem alluded to in § 1 h. This problem is to show that each equivalence $[f]$ in $L_2(\mathbb{R}^p)$ has a representative F that possesses the so-called kth trace in $L_2(\mathbb{R}^{p-2k})$ for each $k \in [0, [p/2]]$. A preliminary result is the following:

13.18. Corollary. (On lifting from $L_2(\mathbb{R}^p)$ to $\mathcal{P}_{1,\xi_p}$) *Let $\forall x \in \mathcal{L}_2$ & $\forall p \in \mathbb{N}_+$,* (i) *$f_x^p$ be the Radon–Nikodym derivative defined in corollary 11.2, and* (ii) *$[f_x^p]$ be the equivalence class in $L_2^{\text{sym}}(\mathbb{R}^p)$ containing f_x^p, cf. 11.1*(*b*)*. Then*

$$\forall x \in \mathcal{L}_2 \quad \& \quad \forall p \in \mathbb{N}_{0+}, \qquad [f_x^p] \cap \mathcal{P}_{1,\xi_p} \neq \emptyset,$$

i.e. for each $x \in \mathcal{L}_2$, $[f_x^p]$ has a representative ϕ_x in $\mathcal{P}_{1,\xi_p}$.

Proof. Let $x \in \mathcal{L}_2$, $p \in \mathbb{N}_+$ & $\hat{x} := \operatorname{proj}(x|\mathcal{S}_{\xi_p})$. Since by 13.17, $\mathcal{S}_{\xi_p} = \operatorname{Range} \mathbb{E}_{\xi_p}$, therefore $\exists F_x \in \mathcal{P}_{1,\xi_p}$ such that $\hat{x} = \mathbb{E}_{\xi_p}(F_x)$. Now since $\mathcal{S}_{\eta_p} \subseteq \mathcal{S}_{\xi_p}$, therefore

(1) $\operatorname{proj}(x|\mathcal{S}_{\eta_p}) = \operatorname{proj}(\hat{x}|\mathcal{S}_{\eta_p}) = \operatorname{proj}\{\mathbb{E}_{\xi_p}(F_x)|\mathcal{S}_{\eta_p}) = \mathbb{E}_{\eta_p}(F_x)$, by 13.16($b$).

Since by theorem 11.1(c), LHS(1) $= (1/p!)\mathbb{E}_{\eta_p}(f_x^p) = \mathbb{E}_{\eta_p}((1/p!)f_x^p)$, we see from (1) that

$$\mathbb{E}_{\eta_p}((1/p!)f_x^p - F_x) = 0, \quad \text{i.e.} \quad (1/p!)f_x^p - F_x \in \text{Null space } (\mathbb{E}_{\eta_p}).$$

Since by 11.1(b), $f_x^p \in L_2^{\text{sym}}(\mathbb{R}^p)$, therefore by theorem 10.5(b),

$$0 = ((1/p!)f_x^p - F_x)\tilde{} = (1/p!)\tilde{f}_x^p - \tilde{F}_x = (1/p!)f_x^p - \tilde{F}_x.$$

Hence $\phi_x^p := p!\tilde{F}_x = f_x^p$ in $L_2(\mathbb{R}^p)$. And since F_x is in $\mathcal{P}_{1,\xi_p}$, so by (13.7) is $\tilde{F}_x$ and therefore so is ϕ_x^p. Thus $\phi_x^p \in \mathcal{P}_{1,\xi_p} \cap [f_x^p]$. ■

13.19. *Remarks.* (The Feynman integral) With the notation of the last corollary the function $\phi_x^p := p!\tilde{F}_x$, being in $\mathcal{P}_{1,\xi_p}$, satisfies the conditions of 13.12, i.e. $\forall k \in [0, [p/2]]$, $(\phi_x^p)_k^p$ exists and is in $\mathcal{P}_{1,\eta_{p-2k}}$. Moreover, since ϕ_x^p is symmetric, therefore by proposition 12.18(e), for ℓ_{p-2k} almost all h in $\mathbb{R}^{p-2k}$,

$$(\phi_x^p)_k^p(h) = \binom{p}{2k}\alpha_{2k}\int_{\mathbb{R}^k} \phi_x^p(\tau_1, \tau_1, \ldots, \tau_k\tau_k; h)\ell_k(\mathrm{d}\tau).$$

The last term, apart from a well-determined constant factor, matches formula (1.3) in Johnson & Kallianpur (1993). The latter formula can thus be secured by the following recipe: Project the given x in $\mathcal{L}_2$ on the subspace $\mathcal{S}_{\xi_p}$ generated by the pth Wiener chaotic measure ξ_p. This, by theorem 13.17, yields an integrand $F_x \in \mathcal{P}_{1,\xi_p}$. Take the symmetrization of this F_x and multiply it by $p!$.

By hindsight, however, we can be even more explicit. *We can avoid all reference to vector measures, and rely instead entirely on theorem 12.17, i.e. on scalar concepts emanating from the diagonal skeletons.* To get from the equivalence class $[f]$, where $f \in L_2(\mathbb{R}^p)$, a representative F for which the 'kth trace' is a given $G_k \in L_2(\mathbb{R}^{p-2k})$,

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$\forall k \in [1, [p/2]]$, take the function, F given by

$$F(t) = \sum_{j=0}^{[p/2]} \rho_j\{\wp_{\pi_k^*}(t)\}G_j(t_{2j+1}, \ldots, t_p)\chi_{I(p,\pi_j)\cap(\mathbb{R}^{2j}\times\mathbb{R}_*^{p-2k})}(t), \qquad \forall t \in \mathbb{R}^p,$$

where each ρ_j is any probability density on $\mathbb{R}^j$, and $G_0 = f$.

Finally, towards establishing contact with Ito's integral, we ask for what kind of f is $\mathbb{E}_{\xi_p}(f) = \mathbb{E}_{\eta_p}(f)$? An answer is provided by the next lemma.

13.20. Lemma. *Let $p \in \mathbb{N}_+$ & $f \in L_2(\mathbb{R}^p)$. Then*

$$\forall f \in \mathcal{P}_{1,\xi_p} \quad \textit{such that} \quad \operatorname{supp} f \subseteq \mathbb{R}_*^p := \mathbb{R}^p \backslash I_1^p, \quad \mathbb{E}_{\xi_p}(f) = \mathbb{E}_{\eta_p}(f).$$

Proof. Case 1. Let $p = 1$. Then $\xi_1 = \eta_1$, and the result holds trivially.

Case 2. Let $p \geqslant 2$, $f \in \mathcal{P}_{1,\xi_p}$ and $\operatorname{supp} f \subseteq \mathbb{R}_*^p$. Then by 12.14($e$), $\forall k \in [1, [p/2]]$, $0 = M_k^p(f) = f_k^p$. Hence by theorem 13.12,

$$\mathbb{E}_{\xi_p}(f) = \sum_{k=0}^{[p/2]} \mathbb{E}_{\eta_{p-2k}}(f_k^p) = \mathbb{E}_{\eta_p}(f_0^p) = \mathbb{E}_{\eta_p}(f),$$

cf. theorem 12.14(f). ■

The Ito integral. Let $p \in \mathbb{N}_+$. Ito (1951) calls a function f in $\mathcal{S}(\mathcal{P}_p, \mathbb{R})$ *special* iff f vanishes on the diagonal skeleton I_1^p. Ito shows that the class Σ_p, of special functions is a linear manifold everywhere dense in $L_2(\mathbb{R})$. Ito's definition of the integral $\underline{I}_p$ reads simply:

$$\forall f \in \Sigma_p, \qquad \underline{I}_p(f) := \mathbb{E}_{\xi_p}(f).^{11}$$

It is shown that $\underline{I}_p$ is a linear operator on Σ_p into $\mathcal{L}_2$ and that $(1/\sqrt{p!})\underline{I}_p$ is a contraction, i.e.

$$\forall f \in \Sigma_p, \qquad |\underline{I}_p(f)| \leqslant \sqrt{p!}\,|f|_{2,\ell_p}.$$

This allows Ito to define $\underline{I}_p(f)$ for any $f \in L_2(\mathbb{R}^p)$, by taking $f_n \in \Sigma_p$ such that $|f - f_n|_{2,\ell_p} \to 0$, and letting

$$\underline{I}_p(f) := \lim_{n\to\infty} \underline{I}_p(f_n).$$

We contend that $\underline{I}_p$ is just $\mathbb{E}_{\eta_p}$:

13.21. Proposition. $\forall f \in L_2(\mathbb{R}^p)$, $\underline{I}_p(f) = \mathbb{E}_{\eta_p}(f)$.

Proof. Let $f \in \Sigma_p$. Then by definition, $\underline{I}_p(f) = \mathbb{E}_{\xi_p}(f)$. But by lemma 13.20, the last term is $\mathbb{E}_{\eta_p}(f)$, since $\operatorname{supp} f \subseteq \mathbb{R}_*^p$. Thus

$$(1) \qquad \forall f \in \Sigma_p, \qquad \underline{I}_p(f) = \mathbb{E}_{\eta_p}(f).$$

Now let $f \in L_2(\mathbb{R}^p)$ and $f_n \in \Sigma_p$, be such that $|f - f_n|_{2,\ell_p} \to 0$. Then by the definition of $\underline{I}_p$, and (1),

$$(2) \qquad \underline{I}_p(f) := \lim_{n\to\infty} \underline{I}_p(f_n) = \lim_{n\to\infty} \mathbb{E}_{\eta_p}(f_n).$$

But by 10.3(b), $|\mathbb{E}_{\eta_p}(f) - \mathbb{E}_{\eta_p}(f_n)| \leqslant \sqrt{p!}\,|f - f_n|_{2,\ell_p} \to 0$. Hence the last limit in (2) is $\mathbb{E}_{\eta_p}(f)$. Thus $\underline{I}_p(f) = \mathbb{E}_{\eta_p}(f)$. ■

[11] The bar under I is inserted to avoid possible confusion with the diagonal skeleton.

14. The Fubini theorem for tensor products of functions on $\mathbb{R}^p$ and on $\mathbb{R}^q$

Let $\mathcal{C}$, $\mathcal{D}$ be δ-rings over some sets S and T, and $\rho \in \mathrm{CA}(\mathcal{C}, \mathcal{L}_2)$, $\sigma \in \mathrm{CA}(\mathcal{D}, \mathcal{L}_2)$. As remarked in 5.24, $\rho \times \sigma \notin \mathrm{CA}\{\delta\text{-ring}(\mathcal{C} \times \mathcal{D}), \mathcal{L}_2\}$, in general.

Now let $f \in \mathcal{M}(\mathcal{C}^{\mathrm{loc}}, \mathcal{B}_1)$, $g \in \mathcal{M}(\mathcal{D}^{\mathrm{loc}}, \mathcal{B}_1)$. Then the (tensor) product $f \times g$ on $S \times T$ defined by

$$(f \times g)(s,t) := f(s) \cdot g(t), \qquad (s,t) \in S \times T,$$

is in $\mathcal{M}[\{\delta\text{-ring}(\mathcal{C} \times \mathcal{D})\}^{\mathrm{loc}}, \mathcal{B}_1]$. Even granting that we have

$$(1) \qquad \rho \times \sigma \in \mathrm{CA}(\delta\text{-ring}(\mathcal{C} \times \mathcal{D}), \mathcal{L}_2),$$

it is *not* clear that in general

$$(2) \qquad f \in \mathcal{P}_{1,\rho} \quad \& \quad g \in \mathcal{P}_{1,\sigma} \Longrightarrow f \times g \in \mathcal{P}_{1,\rho \times g}.$$

The difficulty lies in the absence of a nexus between the Pettis norms $|f \times g|_{1,\rho\times\sigma}$ and $|f|_{1,\rho}$, $|g|_{1,\sigma}$, cf. (A.9). This absence stems from the fact that for X_1, X_2, $Y \in \mathcal{L}_2$, we know of no way to bring about an inequality such as

$$|(X_1 \cdot X_2, Y)| \leqslant |(X_1, Y_1)| \cdot |(X_2, Y_2)|$$

for suitable Y_1, Y_2 in $\mathcal{L}_2$. Nor is there any convenient relationship between the norms $|X_1 \cdot X_2|_{\mathcal{L}_2}$ and $|X_1|_{\mathcal{L}_2}$, $|X_2|_{\mathcal{L}_2}$, that offers an alternative approach.

However, once these two difficult points are conceded, and (1), (2) are taken as premises, the Fubini equality $\mathbb{E}_{\rho\times\sigma}(f \times g) = \mathbb{E}_\rho(f) \cdot \mathbb{E}_\sigma(g)$ is provable by standard considerations, as shown in the next proposition:

14.1. Proposition. (A Fubini theorem for $\mathcal{L}_2$-valued measures) *Let*
(i) $\mathcal{C}$, $\mathcal{D}$ *be* δ*-rings over spaces* S, T & $\hat{D} = \delta\text{-ring}(\mathcal{C} \times \mathcal{D})$,
(ii) $\rho \in \mathrm{CA}(\mathcal{C}, \mathcal{L}_2)$, $\sigma \in \mathrm{CA}(\mathcal{D}, \mathcal{L}_2)$ *be such that* $\rho \times \sigma \in \mathrm{CA}(\hat{D}, \mathcal{L}_2)$,
(iii) $f \in \mathcal{P}_{1,\rho}$, $g \in \mathcal{P}_{1,\sigma}$ *be such that* $f \times g \in \mathcal{P}_{1,\rho\times\sigma}$.
Then

$$\mathbb{E}_{\rho\times\sigma}(f \times g) = \mathbb{E}_\rho(f) \cdot \mathbb{E}_\sigma(g).$$

Proof. Case 1. Let $f \in S(\mathcal{C}, \mathbb{R})$ & $g \in S(\mathcal{D}, \mathbb{R})$, cf. (A.12). Then obviously $f \times g \in S(\hat{D}, \mathbb{R})$, and an elementary computation shows that

$$(1) \qquad \mathbb{E}_{\rho\times\sigma}(f \times g) = \mathbb{E}_\rho(f) \cdot \mathbb{E}_\sigma(g).$$

Case 2. Let $f \in \mathcal{P}_{1,\rho}$ & $g \in \mathcal{P}_{1,\sigma}$ be such that $f \times g \in \mathcal{P}_{1,\rho\times\sigma}$. Then by the approximation theorem A.24,

$$(2) \qquad \begin{cases} \exists (f_n)_1^\infty \in S(\mathcal{C}, \mathbb{R}) \ni |f_n(\cdot)| \leqslant |f(\cdot)|, \qquad |f_n - f|_{1,\rho} \to 0, \\ \& \\ \exists N_1 \in \mathcal{N}_\rho \ni \forall x \in \Omega \backslash N_1, \qquad f_n(x) \to f(x); \end{cases}$$

$$(3) \qquad \begin{cases} \exists (g_n)_1^\infty \in S(\mathcal{D}, \mathbb{R}) \ni |g_n(\cdot)| \leqslant |g(\cdot)|, \qquad |g_n - g|_{1,\sigma} \to 0, \\ \& \\ \exists N_2 \in \mathcal{N}_\sigma \ni \forall y \in \Lambda \backslash N_2, \qquad g_n(y) \to g(y). \end{cases}$$

It follows from (2) and (3), cf. A.26, that

(4) $$\mathbb{E}_\rho(f) := \lim_{n\to\infty} \mathbb{E}_\rho(f_n) \quad \& \quad \mathbb{E}_\sigma(g) = \lim_{n\to\infty} \mathbb{E}_\sigma(g_n).$$

We now assert that

(I) $$\mathbb{E}_{\rho\times\sigma}(f\times g) = \lim_{n\to\infty} \mathbb{E}_{\rho\times\sigma}(f_n \times g_n).$$

Proof of (I). By (2) and (3),

$$\forall (s,t)\in S\times T, \quad |(f_n\times g_n)(s,t)| = |f_n(s)|\cdot|g_n(t)| \leqslant |f(s)|\cdot|g(t)| = |(f\times g)(s,t)|,$$

i.e. by (iii) and (A.11)(*b*),

(5) $$\forall n\in\mathbb{N}_+, \qquad |(f_n\times g_n)(\cdot,\cdot)| \leqslant |(f\times g)(\cdot,\cdot)| \in \mathcal{P}_{1,\rho\times\sigma}.$$

Also from (2) and (3)

$$\forall (s,t)\in (S\backslash N_1)\times(T\backslash N_2),$$

(6) $$(f_n\times g_n)(s,t) = f_n(s)g_n(t) \to f(s)g(t) = (f\times g)(s,t).$$

Letting $N := (N_1\times T)\cup(S\times N_2)$, we know (cf. A.44) that

$$(S\backslash N_1)\times(T\backslash N_2) = (S\times T)\backslash N \quad \& \quad N\in\mathcal{N}_{\rho\times\sigma}.$$

Hence from (6)

(7) $$\forall (s,t)\in(S\times T)\backslash N, \qquad (f_n\times g_n)(s,t)\to(f\times g)(s,t).$$

It follows from (5), (7) and the dominated convergence theorem A.28 that

(8) $$\lim_{n\to\infty}\mathbb{E}_{\rho\times\sigma}(f_n\times g_n) = \mathbb{E}_{\rho\times\sigma}(f\times g).$$

Thus (I).

Let us denote by F_n, F, G_n, G, Φ_n, Φ the random variables

$$\mathbb{E}_\rho(f_n), \quad \mathbb{E}_\rho(f), \quad \mathbb{E}_\sigma(g_n), \quad \mathbb{E}_\sigma(g), \quad \mathbb{E}_{\rho\times\sigma}(f_n\times g_n), \quad \mathbb{E}_{\rho\times\sigma}(f\times g)$$

in $\mathcal{L}_2 := L_2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R})$. Then by case 1,

(9) $$\Phi_n = F_n\cdot G_n,$$

and by (4) and (I),

$$F_n\to F, \quad G_n\to G \quad \& \quad \Phi_n\to\Phi \quad \text{in} \quad \mathcal{L}_2.$$

By three applications of the subsequence principle, we arrive at a $\mathbb{P}$-negligible set $N\subseteq\Omega$, and a sequence $(n_k)_{k=1}^\infty$ in $\mathbb{N}_+$ such that

(10) $$\begin{cases} \forall\omega\in\Omega\backslash N, \\ F_{n_k}(\omega)\to F(\omega), \quad G_{n_k}(\omega)\to G(\omega) \quad \& \quad \Phi_{n_k}(\omega)\to\Phi(\omega) \quad \text{as} \quad k\to\infty. \end{cases}$$

Thus by (10), $\forall\omega\in\Omega\backslash N$,

$$\begin{aligned} F(\omega)G(\omega) &= \lim_{k\to\infty} F_{n_k}(\omega)\lim_{k\to\infty} G_{n_k}(\omega) = \lim_{k\to\infty}[F_{n_k}(\omega)G_{n_k}(\omega)] \\ &= \lim_{k\to\infty}\Phi_{n_k}(\omega), \qquad \text{by (9)} \\ &= \Phi(\omega), \qquad \text{by (10)}. \end{aligned}$$

Thus $\Phi = F \cdot G$ a.e. $\mathbb{P}$ on Ω, and hence in $\mathcal{L}_2$. More fully,

$$\mathbb{E}_{\rho\times\sigma}(f\times g) = \mathbb{E}_\rho(f)\cdot\mathbb{E}_\sigma(g), \qquad \text{a.e. } \mathbb{P} \text{ on } \Omega.$$

■

Our first goal in this section is to show that the measures ξ_p and ξ_q, for which, as we already know from 5.14, $\xi_p \times \xi_q \in \mathrm{CA}(\mathcal{D}_{p+q}, \mathcal{L}_2)$, are again non-pathological in that the implication

$$(*) \qquad f \in \mathcal{P}_{1,\xi_p} \quad \& \quad g \in \mathcal{P}_{1,\xi_q} \Longrightarrow f\times g \in \mathcal{P}_{1,\xi_{p+q}},$$

prevails, and that therefore by 14.1, for $f \in \mathcal{P}_{1,\xi_p}$ & $g \in \mathcal{P}_{1,\xi_q}$, we have both

$$f\times g \in \mathcal{P}_{1,\xi_{p+q}} \quad \& \quad \mathbb{E}_{\xi_{p+q}}(f\times g) = \mathbb{E}_{\xi_p}(f)\cdot\mathbb{E}_{\xi_q}(g).$$

Our second goal will be to find the more difficult connection between the integrations $\mathbb{E}_{\eta_p\times\eta_q}$ and $\mathbb{E}_{\eta_{p+q}}$. The reader willing to take $(*)$, i.e. 14.10 below, on faith can turn to 14.11.

To turn to the implication $(*)$, our ignorance of the connection between the three Pettis norms cited at the outset, and between the three $\mathcal{L}_2$ norms $|X\cdot Y|_{\mathcal{L}_2}$, $|X|_{\mathcal{L}_2}$, $|Y|_{\mathcal{L}_2}$, precludes us from proving that $f\times g \in \mathcal{P}_{1,\xi_{p+q}}$ by a limiting argument starting with simple functions. We are obliged to fall back on the integrability conditions given in 13.12 It is convenient to restate this theorem for a function F on $\mathbb{R}^{p+q}$, in a format which in the special case $F = f\times g$, with $f \in \mathcal{P}_{1,\xi_p}$, $g \in \mathcal{P}_{1,\xi_q}$, will reveal the conditions that $f\times g$ must be shown to satisfy in order that $f\times g \in \mathcal{P}_{1,\xi_{p+q}}$.

14.2. Lemma. *Let $p, q \in \mathbb{N}_+$. Then $F \in \mathcal{P}_{1,\xi_{p+q}}$ iff*

$(\alpha) \qquad F \in L_2(\mathbb{R}^{p+q})$

&

$(\beta) \qquad \forall r \in [1, [\tfrac{1}{2}(p+q)]] \quad \& \quad \forall \pi \in \Pi_r^{p+q},$

$$\int_{\mathbb{R}^{p+q-2r}} \left\{ \int_{\mathbb{R}^r} |F_\pi^{p+q}(\tau,h)|\ell_r(\mathrm{d}\tau) \right\}^2 \ell_{p+q-2r}(\mathrm{d}h) < \infty.$$

The condition (α) is easily verified for the tensor product $f\times g$. For if $f \in \mathcal{P}_{1,\xi_p}$, and $g \in \mathcal{P}_{1,\xi_q}$, then, cf. (13.2), $f \in L_2(\mathbb{R}^p)$, $g \in L_2(\mathbb{R}^q)$, and therefore $f\times g \in L_2(\mathbb{R}^{p+q})$; thus

$$(14.3) \qquad f \in \mathcal{P}_{1,\xi_p} \quad \& \quad g \in \mathcal{P}_{1,\xi_q} \Longrightarrow f\times g \in L_2(\mathbb{R}^{p+q}).$$

As for the condition (β), we have to attend to the sectioning $F_\pi^{p+q}(\tau,h)$ for $F := f\times g$, with f on $\mathbb{R}^p$ to $\mathbb{R}$, g on $\mathbb{R}^q$ to $\mathbb{R}$, and where $\pi \in \Pi_r^{p+q}$, $\tau \in \mathbb{R}^r$ and $h \in \mathbb{R}^{p+q-2r}$. We claim that there always is a factorization

$$(\#) \qquad (f\times g)_\pi^{p+q}(\tau,h) = f_{\pi_1}^p(\tau^1,\hat{h}^1)\cdot g_{\pi_2}^q(\tau^2,\hat{h}^2).$$

This is proved in 14.9(a), which in turn leads to the integral factorization in 14.9(b). This gives us enough control over the integrals in 14.2(β) to allow us to deduce $(*)$ in 14.10.

To find out what the far from obvious π_1, π_2, τ^1, $\hat{h}^1$, τ^2, $\hat{h}^2$ in $(\#)$ might be, it is worth considering an example:

14.4. *Example.* Let $p = 11$, $q = 9$, $r = 7$, and so $p+q-2r = 6$,

$$(1) \qquad \pi = \{\{1,8\},\{2,16\},\{4,6\},\{5,20\},\{9,10\},\{14,19\},\{17,18\}\} \in \Pi_7^{20};$$

say $\pi = \{\Delta_1, \ldots, \Delta_7\}$, $\Delta_\alpha = \{i_\alpha, j_\alpha\}$, $\alpha \in [1,7]$. Then

(2) $$M'_\pi := [1,20]\backslash M_\pi = \{3,7,11,12,13,15\}.$$

Let

$$\tau = (\tau_1, \ldots, \tau_7) \in \mathbb{R}^7 \quad \& \quad h = (h_1, \ldots, h_6) \in \mathbb{R}^6.$$

Then

(3) $$\theta^{20}_{\pi,h}(\tau) = (t_1, \ldots, t_{11}; t_{12}, \ldots, t_{20}),$$

where by 12.2(a), $\forall\alpha \in [1,7]$, $t_{i_\alpha} = \tau_\alpha = t_{j_\alpha}$ & $\forall\beta \in [1,6]$, $t_{m_\beta} = h_\beta$. It follows that

$$t_1 = t_8 = \tau_1, \quad t_2 = t_{16} = \tau_2, \quad t_3 = h_1, \quad t_4 = t_6 = \tau_3,$$

$$t_5 = t_{20} = \tau_4, \quad t_7 = h_2, \quad t_9 = t_{10} = \tau_5, \quad t_{11} = h_3,$$

$$t_{12} = h_4, \quad t_{13} = h_5, \quad t_{14} = t_{19} = \tau_6, \quad t_{15} = h_6, \quad t_{17} = t_{18} = \tau_7.$$

Thus

(4) $$\theta^{20}_{\pi,h}(\tau) = (\tau_1, \tau_2, h_1, \tau_3, \tau_4, \tau_3, h_2, \tau_1, \tau_5, \tau_5, h_3; h_4, h_5, \tau_6, \tau_6, \tau_2, \tau_7, \tau_7, \tau_6, \tau_4).$$

Now let π_1 comprise the cells of π that fall in $[1,11]$, i.e. let

(5) $$\pi_1 := \{\{1,8\}, \{4,6\}, (9,10\}\} \in \Pi^{11}_3.$$

Then, as the reader can check

(6) $$(\tau_1, \tau_2, h_1, \tau_3, \tau_4, \tau_3, h_2, \tau_1, \tau_5, \tau_5, h_3) = \theta^{11}_{\pi_1,\hat{h}^1}(\tau^1),$$

where

(7) $$\tau^1 := (\tau_1, \tau_3, \tau_5) \in \mathbb{R}^3,$$

these being the only τ's that appear twice in the first 11 terms of the sequence (4), and

(8) $$\hat{h}^1 := (\tau_2, h_1, \tau_4, h_2, h_3) \in \mathbb{R}^5 = \mathbb{R}^{11-2\cdot 3},$$

these being the terms still left among the first 11 terms of the sequence (4), after the removal of all the τ's in (7).

Next, take the cells of π that fall in $[12,20]$, namely, $\{14,19\}$, $\{17,18\}$, and displace them by -11, and so obtain the partition

(5′) $$\pi_2 := \{\{3,8\}, \{6,7\}\} \in \Pi^9_2.$$

Then, as the reader can check,

(6′) $$(h_4, h_5, \tau_6, \tau_2, \tau_7, \tau_7, \tau_6, \tau_4) = \theta^9_{\pi_2,\hat{h}^2}(\tau^2),$$

where

(7′) $$\tau^2 = (\tau_6, \tau_7) \in \mathbb{R}^2,$$

these being the only τ's that appear twice in the last nine terms of the sequence (4), and

(8′) $$\hat{h}^2 := (h_4, h_5, h_6, \tau_2, \tau_4) \in \mathbb{R}^5 = \mathbb{R}^{9-2\cdot 2},$$

these being the terms still left among the last nine terms of the sequence (4), after the removal of all the τ's in (7′). On combining (4), (6) and (6′), we see that

(9) $$\theta^{20}_{\pi,h}(\tau) = (\theta^{11}_{\pi_1,\hat{h}^1}(\tau_1), \theta^{11}_{\pi_2,\hat{h}^2}(\tau^2)).$$

Now let f, g be functions on $\mathbb{R}^{11}$ and $\mathbb{R}^9$. Then

$$\begin{aligned}(f\times g)^{20}_{\pi}(\tau,h) &:= (f\times g)\{\theta^{20}_{\pi,h}(h)\} && \text{by 12.2}(b)\\ &= f\{\theta^{11}_{\pi_1,\hat h_1}(\tau^1)\}\cdot g\{\theta^{11}_{\pi_2,\hat h_2}(\tau^2)\}, && \text{by (9)}\\ &=: f^{11}_{\pi_1}(\tau^1,\hat h^1)\cdot g^{9}_{\pi_2}(\tau^2,\hat h^2), && \text{by 12.2}(b).\end{aligned}$$

■

It seems clear from an inspection of example 14.4, that the recipes in it for procuring the ingredients π_1, τ^1, $\hat h^1$ and π_2, τ^2, $\hat h^2$ needed for the factorization, should work in general. This leads us to define these ingredients by means of the recipes themselves:

14.5. *Definition.* (*a*) Let (i) $p,q\in\mathbb{N}_+$, $q\leqslant p$ & $r\in[0,[\frac{1}{2}(p+q)]]$, (ii) $\pi=\{\Delta_1,\dots,\Delta_r\}\in\Pi^{p+q}_r$, $M'_\pi := [1,p+q]\backslash M_\pi$,

$$\forall\alpha\in[1,r],\quad \Delta_\alpha := \{i_\alpha, j_\alpha\},\quad i_1<\cdots<i_r\quad \&\quad i_\alpha<j_\alpha,$$

Then we define

$$\begin{aligned}&\pi_0 := \{\Delta:\Delta\in\pi\ \&\ \min\Delta\leqslant p<\max\Delta\},\\ &\pi_1 := \{\Delta:\Delta\in\pi\ \&\ \Delta\subseteq[1,p]\},\quad M'_{\pi_1} := [1,p]\backslash M_{\pi_1},\\ &\pi_2 := \{\Delta-\{p\}:\Delta\in\pi\ \&\ \Delta\subseteq[p+1,p+q]\},\quad M'_{\pi_2} := [1,q]\backslash M_{\pi_2},\\ &A_1 := \{\alpha:\alpha\in[1,r]\ \&\ j_\alpha\leqslant p\},\\ &A_0 := \{\alpha:\alpha\in[1,r]\ \&\ i_\alpha\leqslant p<j_\alpha\},\\ &A_2 := \{\alpha:\alpha\in[1,r]\ \&\ p+1\leqslant i_\alpha\},\\ &k_i := \#(\pi_i),\quad i=0,1,2;\quad p' := \#(M'_\pi\cap[1,p]);\\ &q' := \#(M'_\pi\cap[p+1,p+q]).\end{aligned}$$

(*b*) With (i), (ii) as in (*a*), let (iii) $\tau=(\tau_1,\dots,\tau_r)\in\mathbb{R}^r$. Then we define

$$\tau^i := (\tau_\alpha:\alpha\in A_i),\qquad i=0,1,2.$$

(*c*) With (i), (ii), (iii) as in (*a*), (*b*), let (iv) $h=(h_1,\dots,h_{p+q-2r})\in\mathbb{R}^{p+q-2k}$. Then we define $h^1 := (h_1,\dots,h_{p'})$, $h^2 := (h_{p'+1},\dots,h_{p'+q'})$.

(*d*) With (i)–(iv) as in (*a*), (*b*), (*c*), let

$$\theta^{p+q}_{\pi,h}(\tau) := (t_1,\dots,t_p;t_{p+1},\dots,t_{p+q}).$$

Then we define

$\hat h^1 :=$ the subsequence of $(t_1,\dots,t_p)$ obtained by deleting from it all terms equal to τ_α, for $\alpha\in A_1$;

$\hat h^2 :=$ the subsequence of $(t_{p+1},\dots,t_{p+q})$ obtained by deleting from it all terms equal to τ_α, for $\alpha\in A_2$.

We shall call π_0, π_1, π_2 and τ^0, τ^1, τ^2 and $\hat h^1$, $\hat h^2$ the *p, q canonical components* of π and τ and h, respectively.

The following connections obviously obtain:

14.6. Triviality. *With the notation 14.5, we have* (*a*) $\pi_1 = \{\Delta_\alpha : \alpha \in A_1\} \in \Pi^p_{k_1}$, $\pi_2 = \{\Delta_\alpha - \{p\} : \alpha \in A_2\} \in \Pi^q_{k_2}$,

$$\pi = \pi_1 \cup \pi_0 \cup \pi_2 + \{p\}, \qquad \pi_0, \pi_1, \pi_2 + \{p\} \quad \textit{are} \quad \|;$$

(*b*) $r = k_0 + k_1 + k_2$, $k_1 \in [0, [p/2]]$, $k_2 \in [0, [q/2]]$;
(*c*) $h = (h^1, h^2)$, $p + q - 2r = p' + q'$;
(*d*) *the vector* $\hat{h}^i$ *is made up of the components of* τ_0 & h^i, $i = 1, 2$;
(*e*) $p - 2k_1 = k_0 + p'$, $q - 2k_2 = k_0 + q'$.

Guided by the example 14.4, we now assert

14.7. Decomposition lemma. *Let*
(i) $p, q \in \mathbb{N}_+$, $q \leqslant p$, $r \in [1, [(p+q)/2]]$, $\pi \in \Pi^{p+q}_r$, $\tau \in \mathbb{R}^r$ & $h \in \mathbb{R}^{p+q-2r}$;
(ii) $\pi_0, \pi_1, \pi_2, \tau^0, \tau^1, \tau^2$ & $\hat{h}^1, \hat{h}^2$ *be the* p, q *canonical components of* π, τ *and* h, *and* $k_i = \#(\pi_i)$, *for* $i = 0, 1, 2$.
Then

$$\theta^{p+q}_{\pi,h}(\tau) := (\theta^p_{\pi_1,\hat{h}^1}(\tau^1), \theta^q_{\pi_2,\hat{h}^2}(\tau^2)).$$

Proof. Let

$$\theta^{p+q}_{\pi,h}(\tau) := (t_1, \ldots, t_p; t_{p+1}, \ldots, t_{p+q}). \tag{1}$$

Then we have only to show that

$$\text{(I)} \qquad (t_1, \ldots, t_p) = \theta^p_{\pi_1,\hat{h}^1}(\tau^1); \qquad \text{(II)} \qquad (t_{p+1}, \ldots, t_{p+q}) = \theta^q_{\pi_2,\hat{h}^2}(\tau^2).$$

Proof of (I). Let

$$\theta^p_{\pi_1,\hat{h}^1}(\tau^1) = (s_1, \ldots, s_p); \tag{2}$$

$$M'_\pi := [1, p+q] \backslash M_\pi = \{m_1, \ldots, m_{p+q-2r}\}, \quad m_1 < \cdots < m_{p+q-2r}; \tag{3}$$

$$M'_{\pi_1} := [1, p] \backslash M_{\pi_1} = \{n_1, \ldots, n_{p-2k_1}\}, \quad n_1 < \cdots < n_{p-2k_1}. \tag{4}$$

Then by 12.2 and (1), (2),

$$\forall \alpha \in [1, r], \quad t_{i_\alpha} = \tau_\alpha = t_{j_\alpha} \quad \& \quad \forall \beta \in [1, p+q-2r], \quad t_{m_\beta} = h_\beta; \tag{5}$$

$$\forall \alpha \in A_1, \quad s_{i_\alpha} = \tau_\alpha = s_{j_\alpha} \quad \& \quad \forall \gamma \in [1, p-2k_1], \quad s_{n_\gamma} = \hat{h}^1_\gamma. \tag{6}$$

Now let $\mu \in M_{\pi_1}$. Then, cf. 14.6(*a*), $\mu = i_\alpha$ or $\mu = j_\alpha$, for some $\alpha \in A_1$. Since $\alpha \in A_1$, therefore by (6), $s_{i_\alpha} = \tau_\alpha = s_{j_\alpha}$, $s_\mu = \tau_\alpha$. But by 14.5(*a*), $A_1 \subseteq [1, r]$, and so $\alpha \in [1, r]$, and therefore by (5), $t_{i_\alpha} = \tau_\alpha = t_{j_\alpha}$, i.e. $t_\mu = \tau_\alpha$. Thus s_μ and t_μ are equal to the same τ_α. We thus see that

$$\forall \mu \in M_{\pi_1}, \quad s_\mu = t_\mu. \tag{7}$$

Next, consider any $\mu \in [1, p] \backslash M_{\pi_1}$. By (4), $\mu \in \{n_1, \ldots, n_{p-2k_1}\}$. Now the sequence $(t_{n_1}, \ldots, t_{n_{p-2k_1}})$ is precisely the subsequence of $(t_1, \ldots, t_p)$ that we get on deleting from it, all terms equal to τ_α, for some $\alpha \in A_1$. But this subsequence is by definition, cf. 14.5(*d*), precisely the sequence $\hat{h}^1$. Thus

$$\begin{aligned} (t_{n_1}, \ldots, t_{n_{p-2k_1}}) &= \hat{h}^1 = (\hat{h}^1_1, \ldots, \hat{h}^1_{p-2k_1}) \\ &= (s_{n_1}, \ldots, s_{n_{p-2k_1}}), \qquad \text{by (6).} \end{aligned} \tag{8}$$

Since $\mu \in \{n_1, \ldots, n_{p-2k_1}\}$, therefore $\mu =$ some n_β. Thus by (8), $s_\mu = s_{n_\beta} = t_{n_\beta} = t_\mu$. This shows that

$$\forall \mu \in [1, p] \backslash M_{\pi_1}, \quad s_\mu = t_\mu. \tag{9}$$

Combining (7) and (9), we see that $\forall \mu \in [1,p]$, $s_\mu = t_\mu$. Thus

$$(t_1,\ldots,t_p) = (s_1,\ldots,s_p) = \theta^p_{\pi_1,\hat{h}^1}(\tau^1), \qquad \text{by (2)}.$$

Thus (I).

Proof of (II). This is obtained by an obvious adaptation of the proof of (I). ■

The decomposition lemma yields readily the factorization (#) (see 14.9(*a*) below). However, the usability of this factorization rests on the structure of the vectors $\hat{h}^1$, $\hat{h}^2$ revealed in 14.6(*d*). To exhibit the particular mix of τ^0 and h^i that constitute the vector $\hat{h}^i$, where $i = 1, 2$, we shall adopt the following notation:

(14.8) $$[\tau^0, h^1] := \hat{h}^1, \quad [\tau^0, h^2] := \hat{h}^2.$$

This notation is needed in part (*b*) of the next lemma.

14.9. Factorization lemma. *Let* (i) *and* (ii) *be as in 14.7. Then*

(*a*) $\forall$ *functions* f, g *on* $\mathbb{R}^p$ *and* $\mathbb{R}^q$, *respectively,*

$$(f \times g)^{p+q}_\pi(\tau, h) = f^p_{\pi_1}(\tau^1, \hat{h}^1)\cdot g^q_{\pi_2}(\tau^2, \hat{h}^2);$$

(*b*) $\forall f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$ & $\forall g \in \mathcal{M}(\mathcal{B}_q, \mathcal{B}_1)$,

$$\int_{\mathbb{R}^r} |(f\times g)^{p+q}_\pi(\tau,h)|\ell_r\,(\mathrm{d}\tau) = \int_{\mathbb{R}^{k_0}} \left\{ \left[\int_{R^{k_1}} |f^p_{\pi_1}(\tau^1, [\tau^0, h^1])|\ell_{k_1}\,(\mathrm{d}\tau^1) \right] \cdot \left[\int_{\mathbb{R}^{k_2}} |g^q_{\pi_2}(\tau^2, [\tau^0, h^2])|\ell_{k_2}\,(\mathrm{d}\tau^2) \right] \right\} \ell_{k_0}\,(\mathrm{d}\tau^0) \in [0,\infty].$$

Proof. (*a*) Let f, g be on $\mathbb{R}^p$ and $\mathbb{R}^q$. Then by 12.2, the decomposition lemma and the definition of the tensor product

$$(f\times g)^{p+q}_\pi(\tau,h) := (f\times g)\{\theta^{p+q}_{\pi,h}(\tau)\} = (f\times g)(\theta^p_{\pi_1,\hat{h}^1}(\tau^1), \theta^q_{\pi_2,\hat{h}^2}(\tau^2))$$
$$= f\{\theta^p_{\pi_1,\hat{h}^1}(\hat{\tau}^1)\}\cdot g\{\theta^q_{\pi_2,\hat{h}^2}(\hat{\tau}^2)\} =: f^p_{\pi_1}(\hat{\tau}^1, \hat{h}^1)\cdot g^q_{\pi_2}(\hat{\tau}^2, \hat{h}^2).$$

Thus (*a*).

(*b*) Write $F := f \times g$. Since, cf. 14.5(*b*), each component τ_α of τ, $\alpha \in [1, r]$, falls among the components of τ^0, τ^1, τ^2, therefore $|F^{p+q}_\pi(\cdot, h)|$ is equal to a function $G(\cdot,\cdot,\cdot;h)$:

$$\forall \tau \in \mathbb{R}^r, \quad |F^{p+q}_\pi(\tau,h)| = G(\tau^1,\tau^2,\tau^0;h).$$

Since $r = k_0 + k_1 + k_2$, therefore by Tonelli's theorem,

(1) $$\int_{\mathbb{R}^r} |F^{p+q}_\pi(\tau,h)|\ell_r\,(\mathrm{d}\tau) = \int_{\mathbb{R}^{k_0}} \left[\int_{\mathbb{R}^{k_1+k_2}} G(\tau^1,\tau^2,\tau^0;h)\ell_{k_1+k_2}\{\mathrm{d}(\tau^1,\tau^2\} \right] \ell_{k_0}\,(\mathrm{d}\tau^0).$$

Factoring G as per the equality in (*a*), and applying Tonelli's theorem, the integrand on the RHS(1) is seen to be

$$\int_{\mathbb{R}^{k_1+k_2}} |f^p_{\pi_1}(\tau^1,\hat{h}^1)\cdot g^q_{\pi_2}(\tau^2,\hat{h}^2)|\ell_{k_1+k_2}\{\mathrm{d}(\tau^1,\tau^2)\}$$
$$= \int_{\mathbb{R}^{k_1}} \left\{ \int_{\mathbb{R}^{k_2}} |f^p_{\pi_1}(\tau_1,\hat{h}^1)|\cdot|g^q_{\pi_2}(\tau^2,\hat{h}^2)|\ell_{k_2}\,(\mathrm{d}\tau^2) \right\} \ell_{k_1}\,(\mathrm{d}\tau^2)$$
$$= \int_{\mathbb{R}^{k_1}} |f^p_{\pi_1}(\tau_1,\hat{h}^1)|\ell_{k_1}\,(\mathrm{d}\tau^1)\cdot \int_{\mathbb{R}^{k_2}} |g^q_{\pi_2}(\tau_2,\hat{h}^2)|\ell_{k_2}\,(\mathrm{d}\tau^2).$$

Substituting this expression on the RHS of (1), and substituting for $\hat{h}^1$, $\hat{h}^2$ from (14.8), we get (b). ■

The last lemma allows us to show that the criteria for F to be in $\mathcal{P}_{1,\xi_{p+q}}$, given in 14.2, are fulfilled when $F = f \times g$, and $f \in \mathcal{P}_{1,\xi_p}$ and $g \in \mathcal{P}_{1,\xi_q}$, and thereby to prove:

14.10. Main theorem. *Let $p, q \in \mathbb{N}_+$, $f \in \mathcal{P}_{1,\xi_p}$ and $g \in \mathcal{P}_{1,\xi_q}$. Then* *$(a)$ $f \times g \in \mathcal{P}_{1,\xi_{p+q}}$ and (b) $\mathbb{E}_{\xi_{p+q}}(f \times g) = \mathbb{E}_{\xi_p}(f) \cdot \mathbb{E}_{\xi_q}(g)$.*

Proof. (a) By (14.3), $F := f \times g \in L_2(\mathbb{R}^{p+q})$. Hence we have only to verify the condition in 14.2(β). Let $r \in [1, [(p+q)/2]]$. We have only to show that

$$\forall \pi \in \Pi_r^{p+q}, \quad J := \int_{\mathbb{R}^{p+q-2r}} \left[\int_{\mathbb{R}^r} |F_\pi^{p+q}(\tau, h)| \ell_r\,(\mathrm{d}\tau) \right]^2 \ell_{p+q-2r}\,(\mathrm{d}h) < \infty.$$

Fix $\pi \in \Pi_r^{p+q}$ and let π_0, π_i, π_2 be the p, q canonical components of π. Then $\pi_1 \in \Pi_{k_1}^p$, $\pi_2 \in \Pi_{k_2}^p$. Since $f \in P_{1,\xi_p}$ and $g \in P_{1,\xi_q}$, we know from 14.2(β) that

$$(1) \qquad \begin{cases} J_1 := \displaystyle\int_{\mathbb{R}^{p-2k_1}} \left[\int_{\mathbb{R}^{k_1}} |f_{\pi_1}^p(\tau^1, \hat{h}^1)| \ell_{k_1}\,(\mathrm{d}\tau^1) \right]^2 \ell_{p-2k_1}\,(\mathrm{d}\hat{h}^1) < \infty, \\ J_2 := \int_{\mathbb{R}^{p-2k_2}} \left[\int_{\mathbb{R}^{k_2}} |g_{\pi_2}^q(\tau^2, \hat{h}^2)| \ell_{k_2}\,(\mathrm{d}\tau^2) \right]^2 \ell_{q-2k_2}\,(\mathrm{d}\hat{h}^2) < \infty. \end{cases}$$

We shall complete the proof by showing that

$$(\mathrm{I}) \qquad J \leqslant J_1 \cdot J_2 \quad \text{and so} \quad J < \infty.$$

Proof of (I). Let $\tau^0 \in \mathbb{R}^{k_0}$ and $[\tau^0, h^i]$ be as in 14.8. It then follows from 14.9(b) that

$$(2) \qquad \begin{aligned} \Phi(h) &:= \int_{\mathbb{R}^r} |F_\pi^p(\tau, h)| \ell_r\,(\mathrm{d}\tau) = \int_{\mathbb{R}^r} |(f \times g)_\pi^p(\tau, h)| \ell_r\,(\mathrm{d}\tau) \\ &= \int_{\mathbb{R}^{k_0}} \left\{ \left[\int_{\mathbb{R}^{k_1}} |f_{\pi_1}^p(\tau^1, [\tau^0, h^1])| \ell_{k_1}\,(\mathrm{d}\tau^1) \right] \right. \\ &\qquad \left. \cdot \left[\int_{\mathbb{R}^{k_2}} |g_{\pi_2}^q(\tau^2, [\tau^0, h^2])| \ell_{k_2}\,(\mathrm{d}\tau^2) \right] \right\} \cdot \ell_{k_0}\,(\mathrm{d}\tau^0). \end{aligned}$$

Squaring both sides, and using the Schwartz inequality on the RHS, we get

$$(3) \qquad \Phi(h)^2 \leqslant \psi_1(h^1) \cdot \psi_2(h^2),$$

where

$$(4) \qquad \psi_1(h^1) := \int_{\mathbb{R}^{k_0}} \left[\int_{\mathbb{R}^{k_1}} |f_{\pi_1}^p(\tau^1, [\tau^0, h^1])| \ell_{k_1}\,(\mathrm{d}\tau^1) \right]^2 \ell_{k_0}\,(\mathrm{d}\tau^0),$$

$$(5) \qquad \psi_2(h^2) := \int_{\mathbb{R}^{k_0}} \left[\int_{\mathbb{R}^{k_2}} |g_{\pi_2}^q(\tau^2, [\tau^0, h^2])| \ell_{k_2}\,(\mathrm{d}\tau^2) \right]^2 \ell_{k_0}\,(\mathrm{d}\tau^0).$$

Noting that $p + q - 2r = p' + q'$, cf. 14.6(c), and using (3), we get

$$J := \int_{\mathbb{R}^{p+q-2r}} \Phi(h)^2 \ell_{p+q-2r}\,(\mathrm{d}h) \leqslant \int_{\mathbb{R}^{p'+q'}} \psi_1(h^1) \cdot \psi_2(h^2) \ell_{p'+q'} \{\mathrm{d}(h^1, h^2)\},$$

i.e.

$$(6) \qquad J \leqslant \int_{\mathbb{R}^{p'}} \psi_1(h^1) \ell_{p'}(\mathrm{d}h^1) \cdot \int_{\mathbb{R}^{q'}} \psi_2(h^2) \ell_{q'}\,(\mathrm{d}h^2).$$

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But by (4),

$$\begin{aligned}\int_{\mathbb{R}^{p'}} \psi_1(h^1)\ell_{p'}(\mathrm{d}h^1) \\ &:= \int_{\mathbb{R}^{p'}} \left\{ \int_{\mathbb{R}^{k_0}} \left[\int_{\mathbb{R}^{k_1}} |f^p_{\pi_1}(\tau^1, [\tau^0, h^1])| \ell_{k_1}(\mathrm{d}\tau^1) \right]^2 \ell_{k_0}(\mathrm{d}\tau^0) \right\} \ell_{p'}(\mathrm{d}h^1) \\ &= \int_{\mathbb{R}^{p'+k_0}} \left[\int_{\mathbb{R}^{k_1}} |f^p_{\pi_1}(\tau^1, [\tau^0, h^1])| \ell_{k_1}(\mathrm{d}\tau^1) \right]^2 \ell_{k_0+p'}\{\mathrm{d}(\tau^0, h^1)\}. \qquad (7)\end{aligned}$$

Now $[\tau^0, h^1]$ is the specific mix-up of the components of τ^0 and h^1 that yields $\hat{h}^1 \in \mathbb{R}^{p-2k_1}$. Thus $\exists \phi \in \mathrm{Perm}(p-2k_1)$, which unmixes these components, i.e. is such that

$$(\hat{h}^1)^\phi = [\tau^0, h^1]^\phi = (\tau_0, h^1).$$

Since Lebesgue integration is permutation invariant, it follows that we can replace $\mathrm{d}(\tau^0, h^1)$ by $\mathrm{d}\hat{h}^1$ in the last integral in (7), and recalling that $p' + k_0 = p - 2k_1$, cf. 14.6(e), conclude that

$$\int_{\mathbb{R}^{p'}} \psi_1(h^1)\ell_{p'}(\mathrm{d}h') = \int_{\mathbb{R}^{p-2k_1}} \left[\int_{\mathbb{R}^{k_1}} |f^p_{\pi_1}(\tau^1, \hat{h}_1)| \ell_{k_1}(\mathrm{d}\tau^1) \right]^2 \ell_{p-2k_1}(\mathrm{d}\hat{h}_1) =: J_1.$$

We can similarly show that

$$\int_{\mathbb{R}^{q-q'}} \psi_2(h^2)\ell_{q-q'}(\mathrm{d}h^2) = J_2.$$

Thus (6) reduces to the inequality (I). This establishes (a).

(b) follows from (a) and proposition 14.1. ■

From the last result we readily get, by iteration, the generalization to any finite number of factors:

14.11. Corollary. *Let* $r, p_1, \ldots, p_r \in \mathbb{N}_+$ & $\forall i \in [1, r]$, $f_i \in \mathcal{P}_{1,\xi_{p_i}}$. *Then*

$$f_1 \times \cdots \times f_r \in \mathcal{P}_{1,\xi_{p_1+\cdots+p_r}} \quad \& \quad \mathbb{E}_{\xi_{p_1+\cdots+p_r}}(f_1 \times \cdots \times f_r) = \prod_{i=1}^{r} \mathbb{E}_{\xi_{p_i}}(f_i).$$

A simple corollary of corollary 14.11 is the result that Wiener did not quite prove (cf. Wiener 1938, eqn (80)):

14.12. Corollary. (Wiener's result) *Let* $p \in \mathbb{N}_+$, *and* $f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$ *be such that*

$$\forall t \in (t_1, \ldots, t_p) \in \mathbb{R}^p, \qquad |f(t)| \leqslant \prod_{i=1}^{p} |f_i(t_i)|,$$

where $f_1, \ldots, f_p \in L_2(\mathbb{R})$. *Then* $f \in \mathcal{P}_{1,\xi_p}$.

Proof. By 14.11, $f_1 \times \cdots \times f_p \in \mathcal{P}_{1,\xi_p}$. Since $\forall t \in \mathbb{R}^p$, $|f(t)| \leqslant |(f_1 \times \cdots \times f_p)(t)|$, therefore by the domination principle (A.18), $f \in \mathcal{P}_{1,\xi_p}$. ■

Another interesting corollary of 14.11 is the fact that for f in $\mathcal{P}_{1,\xi_p}$, the integral, $\mathbb{E}_{\xi_p}(f)$, as a random variable, possesses finite raw moments of all orders:

14.13. Corollary. $\forall p \in \mathbb{N}_+$, $\forall f \in \mathcal{P}_{1,\xi_p}$ & $\forall r \in \mathbb{N}_+$, $\mathbb{E}_{\mathbb{P}}(|\mathbb{E}_{\xi_p}(f)|^r) < \infty$, *i.e. the absolute moments of all orders of the random variable* $\mathbb{E}_{\xi_p}(f)$ *are finite.*

Proof. We first observe that since the $\mathcal{L}_r$ spaces involve a probability measure, therefore

(1) $$\mathcal{L}_2 \subseteq \mathcal{L}_1.$$

Now let $p \in \mathbb{N}_+$, $f \in \mathcal{P}_{1,\xi_p}$, $F := \mathbb{E}_{\xi_p}(f)$ on Ω, and $r \in \mathbb{N}_+$. Define $\forall i \in [1,r]$, $p_i = p$ and $f_i = f$. Then by 14.11,

(2) $$f^{\times r} \in \mathcal{P}_{1,\xi_{pr}} \quad \& \quad F(\cdot)^r = [\{\mathbb{E}_{\xi_p}(f)\}(\cdot)]^r = [\mathbb{E}_{\xi_{pr}}(f^{\times r})](\cdot) \quad \text{on} \quad \Omega.$$

Taking the absolute value, and integrating over Ω, we get

(3) $$\mathbb{E}_{\mathbb{P}}(|\mathbb{E}_{\xi_p}(f)(\cdot)|^r) = \mathbb{E}_{\mathbb{P}}(|\{\mathbb{E}_{\xi_{pr}}(f^{\times r})\}(\cdot)|).$$

But since by (2), $f^{\times r} \in \mathcal{P}_{1,\xi_{pr}}$, therefore $\mathbb{E}_{\xi_{pr}}(f^{\times r}) \in \mathcal{L}_2$. By (1), $\mathbb{E}_{\xi_{pr}}(f^{\times r}) \in \mathcal{L}_1$. Hence $\mathbb{E}_{\mathbb{P}}(|\mathbb{E}_{\xi_{pr}}(f^{\times r})|) < \infty$, i.e. by (3), $\mathbb{E}_{\mathbb{P}}(|\mathbb{E}_{\xi_p}(f)(\cdot)|^r) < \infty$. ■

14.14. *Remarks.* (The general Fubini theorem) Let $p, q \in \mathbb{N}_+$ & F on $\mathbb{R}^{p+q}$ be of the form

(1) $$F := \sum_{k=1}^{r} a_k(f_k \times g_k), \quad \text{where} \quad f_k \in \mathcal{P}_{1,\xi_p} \quad \& \quad g_k \in \mathcal{P}_{1,\xi_q}.$$

Then since $\mathcal{P}_{1,\xi_{p+q}}$ is a Banach space and $\mathbb{E}_{\xi_{p+q}}$ a linear operator on $\mathcal{P}_{1,\xi_{p+q}}$ to $\mathcal{L}_2$, it follows readily from theorem 4.10 that

$$\mathbb{E}_{\xi_{p+q}}(F) = \sum_{k=1}^{r} a_k \mathbb{E}_{\xi_p}(f) \cdot \mathbb{E}_{\xi_q}(g),$$

i.e. $\mathbb{E}_{\xi_{p+q}}(F)$ is a sum of products of integral factors. This result can be extended to all F on $\mathbb{R}^{p+q}$, which are limits of sums of the type (1).

However, even for the simplest case, $p = q = 1$, the general Fubini theorem, to wit, $\forall F \in \mathcal{P}_{1,\xi_2}$,

(2) $$\mathbb{E}_{\xi_2}(F) = \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} F(s,t)\xi_1\,(\mathrm{d}s) \right\} \xi_1\,(\mathrm{d}t)$$

goes beyond the scope of the integration theory used in this paper. This is because the integrand in (2), namely, the partial integral $G(\cdot)$ defined on $\mathbb{R}$ by

$$\forall t \in \mathbb{R}, \qquad G(t) = \int_{\mathbb{R}} F(s,t)\xi_1\,(\mathrm{d}s) \in \mathcal{L}_2,$$

is not scalar-valued but random-variable-valued, and consequently

$$\mathrm{RHS}(2) = \int_{\mathbb{R}} G(t)\xi_1\,(\mathrm{d}t)$$

is undefined. The same difficulty afflicts the general slicing equality

$$\xi_{p+q}(D) = \int_{\mathbb{R}^q} \xi_p(D^t)\xi_q\,(\mathrm{d}t).$$

We turn next to our second goal, namely, the integrability and integration with respect to the product measure $\eta_p \times \eta_q$. Since by proposition 11.13,

$$\eta_p \times \eta_q = \eta_{p+q} + \rho,$$

where $\rho(\cdot) := \zeta_{p+q}(\cdot \backslash J_1^{p+q})$ is given by lemma 11.17, we first attend to integrability and integration with respect to the measure ρ.

14.15. Lemma. *Let* (i) $q \in \mathbb{N}_+$ & $p \leqslant q$, *and (with the notation 11.15),*
(ii) $\forall D \in \mathcal{D}_{p+q}$,

$$\rho(D) = \sum_{r=1}^{q} \int_{\mathbb{R}^{p+q-2r}} \left[\sum_{\pi \in \mathring{\Pi}_r^{p+q}} \lambda_\pi^{p+q}(D,h) \right] \eta_{p+q-2r}(\mathrm{d}h).$$

Then
(*a*) $F \in \mathcal{P}_{1,\rho}$ *iff* $F \in \mathcal{M}(\mathcal{B}_{p+q}, \mathcal{B}_1)$ &

$$\sum_{r=1}^{q} \sum_{\pi \in \mathring{\Pi}_r^{p+q}} \int_{\mathbb{R}^{p+q-2r}} \left[\int_{\mathbb{R}^r} |F_\pi^{p+q}(\tau,h)| \ell_r(\mathrm{d}t) \right]^2 \ell_{p+q-2r}(\mathrm{d}h) < \infty;$$

(*b*) $\forall F \in \mathcal{P}_{1,\rho}$,

$$\mathbb{E}_\rho(F) = \sum_{r=1}^{q} \sum_{\pi \in \mathring{\Pi}_r^{p+q}} \int_{\mathbb{R}^{p+q-2r}} \left[\int_{\mathbb{R}^r} F_\pi^{p+q}(\tau,h) \ell_r(\mathrm{d}t) \right] \eta_{p+q-2r}(\mathrm{d}h).$$

Proof. (*a*) Let $D \in \mathcal{D}_{p+q}$. Then by (ii),

(1) $$\rho(D) = \sum_{r=1}^{q} \int_{\mathbb{R}^{p+q-2r}} K_r(D,h) \eta_{p+q-2r}(\mathrm{d}h),$$

where $\forall r \in [1,q]$, $\forall h \in \mathbb{R}^{p+q-2r}$ & $\forall D \in \mathcal{D}_{p+q}$,

(2) $$K_r(D,h) := \sum_{\pi \in \mathring{\Pi}_r^{p+q}} \lambda_\pi^{p+q}(D,h).$$

But, as is easily checked, with $\mathcal{H} = \mathcal{L}_2$ and $\sigma = \eta_q$, $K_r(\cdot,\cdot)$ satisfies the conditions in (B.3) on $\mathcal{D}_{p+q} \times \mathbb{R}^{p+q-2r}$. Hence on letting

(3) $$\forall D \in \mathcal{D}_{p+q}, \quad \rho_r(D) := \int_{\mathbb{R}^{p+q-2r}} K_r(D,h) \eta_{p+q-2r}(\mathrm{d}h) \in \mathcal{S}_{\eta_{p+q-2r}},$$

it follows from lemma B.5(*a*) that each $\rho_r \in \mathrm{CA}(\mathcal{D}_{p+q}, \mathcal{L}_2)$. Let $F \in \mathcal{M}(\mathcal{B}_{p+q}, \mathcal{B}_1)$. Then by theorem B.8(*a*),

(4) $$F \in \mathcal{P}_{1,\rho_r} \iff \int_{\mathbb{R}^{p+q-2r}} \left\{ \int_{\mathbb{R}^{p+q}} |F(t)| K_r(\mathrm{d}t,h) \right\}^2 \ell_{p+q-2r}(\mathrm{d}h) < \infty.$$

But by (2),

$$\begin{aligned} \int_{\mathbb{R}^{p+q}} |F(t)| K_r(\mathrm{d}t,h) &= \sum_{\pi \in \mathring{\Pi}_r^{p+q}} \int_{\mathbb{R}^{p+q}} |F(t)| \lambda_\pi^{p+q}(\mathrm{d}t,h) \\ &= \sum_{\pi \in \mathring{\Pi}_r^{p+q}} \int_{\mathbb{R}^r} |F_\pi^{p+q}(\tau,h)| \ell_r(\mathrm{d}\tau), \qquad \text{by 12.9}(c). \end{aligned}$$

Thus (4) can be rewritten

(5) $$F \in \mathcal{P}_{1,\rho_r} \iff \int_{\mathbb{R}^{p+q-2r}} \left[\sum_{\pi \in \mathring{\Pi}_r^{p+q}} \int_{\mathbb{R}^r} |F_\pi^{p+q}(\tau,h)| \ell_r(\mathrm{d}\tau) \right]^2 \ell_{p+q-2r}(\mathrm{d}h) < \infty.$$

Now from the trivial equalities for positive functions $\alpha_\pi(\cdot)$ on $\mathbb{R}^n$,

$$\sum_{\pi\in\varPi}\alpha_\pi(h)^2 \leqslant \Big\{\sum_{\pi\in\varPi}\alpha_\pi(h)\Big\}^2 \leqslant \#(\varPi)\cdot\sum_{\pi\in\varPi}\alpha_\pi(h)^2,$$

it follows, since $\#(\varPi)<\infty$, that

$$\int_{\mathbb{R}^n}\Big\{\sum_{\pi\in\varPi}\alpha_\pi(h)\Big\}^2\ell_n(\mathrm{d}h)<\infty \quad\Longleftrightarrow\quad \sum_{\pi\in\varPi}\int_{\mathbb{R}^n}\alpha_\pi(h)^2\ell_n(\mathrm{d}h)<\infty.$$

Applying this with $n=p+q-2r$, $\varPi=\mathring{\varPi}_r^{p+q}$ and

$$\alpha_\pi(h)=\int_{\mathbb{R}^r}|F_\pi^{p+q}(\tau,h)|\ell_r(\mathrm{d}\tau),$$

we infer from (5) that

$$(6)\quad F\in\mathcal{P}_{1,\rho_r} \quad\Longleftrightarrow\quad \sum_{\pi\in\mathring{\varPi}_r^{p+q}}\int_{\mathbb{R}^{p+q-2r}}\Big[\int_{\mathbb{R}^r}|F_\pi^{p+q}(\tau,h)|\ell_r(\mathrm{d}\tau)\Big]^2\ell_{p+q-2r}(\mathrm{d}h)<\infty.$$

Now by (1) and (3),

$$(7)\qquad \rho=\sum_{r=1}^{q}\rho_r,\quad \text{and for}\quad r\neq r',\qquad \mathcal{S}_{\rho_r}\subseteq\mathcal{S}_{\eta_{p+q-2r}}\perp\mathcal{S}_{\eta_{p+q-2r'}}\supseteq\mathcal{S}_{\rho_{r'}}.$$

It follows from A.31(b) that

$$F\in\mathcal{P}_{1,\rho}\quad\Longleftrightarrow\quad\forall r\in[1,q],\qquad F\in\mathcal{P}_{1,\rho_r}.$$

Hence by (6)

$$F\in\mathcal{P}_{1,\rho}\quad\Longleftrightarrow\quad\sum_{r=1}^{q}\sum_{\pi\in\mathring{\varPi}_r^{p+q}}\int_{\mathbb{R}^{p+q-2r}}\Big[\int_{\mathbb{R}^r}|F_\pi^{p+q}(\tau,h)|\ell_r(\mathrm{d}\tau)\Big]^2\ell_{p+q-2r}(\mathrm{d}h)<\infty.$$

This proves (a).

(b) Let $F\in\mathcal{P}_{1,\rho}$. Then by (a), $\forall r\in[1,q]$,

$$\sum_{\pi\in\mathring{\varPi}_r^{p+q}}\int_{\mathbb{R}^{p+q-2r}}\Big[\int_{\mathbb{R}^r}|F_\pi^{p+q}(\tau,h)|\ell_r(\mathrm{d}\tau)\Big]^2\ell_{p+q-2r}(\mathrm{d}h)<\infty,$$

i.e. by (6), $F\in\mathcal{P}_{1,\rho_r}$. Thus, $F\in\bigcap_{r=1}^q\mathcal{P}_{1,\rho_r}$, and it follows from (7) that

$$(8)\qquad \mathbb{E}_\rho(F)=\sum_{r=1}^{q}\mathbb{E}_{\rho_r}(F).$$

But $\rho_r(\cdot)$ is defined by (3). Hence by theorem B.8(b),

$$(9)\qquad \mathbb{E}_{\rho_r}(F)=\int_{\mathbb{R}^{p+q-2r}}\Big\{\int_{\mathbb{R}^{p+q}}F(t)K_r(\mathrm{d}t,h)\Big\}\eta_{p+q-2r}(\mathrm{d}h).$$

Next, from (4) it follows that for ℓ_{p+q-2r} almost all $h\in\mathbb{R}^{p+q-2r}$, $F\in L_{1,K_r(\cdot,h)}$,

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whence by (2), $F \in L_{1,\lambda_\pi^{p+q}(\cdot,h)}$ for each $\pi \in \mathring{\Pi}_r^{p+q}$. Hence by (2) and 12.9(*c*),

$$\int_{\mathbb{R}^{p+q}} F(t) K_r(\mathrm{d}t, h) = \sum_{\pi \in \mathring{\Pi}_r^{p+q}} \int_{\mathbb{R}^{p+q}} F(t) \lambda_\pi^{p+q}(\mathrm{d}t, h)$$
$$= \sum_{\pi \in \mathring{\Pi}_r^{p+q}} \int_{\mathbb{R}^r} F_\pi^{p+q}(\tau, h) \ell_r(\mathrm{d}\tau).$$

Thus the equality (9) reduces to

(10) $$\mathbb{E}_{\rho_r}(F) = \int_{\mathbb{R}^{p+q-2r}} \left[\sum_{\pi \in \mathring{\Pi}_r^{p+q}} \int_{\mathbb{R}^r} F_\pi^{p+q}(\tau, h) \ell_r(\mathrm{d}\tau) \right] \eta_{p+q-2r}(\mathrm{d}h).$$

Combining (8) and (10), we get (*b*). ∎

With the last lemma in place, it is easy to demarcate both $\mathcal{P}_{1,\eta_p \times \eta_q}$ and $\mathbb{E}_{\eta_p \times \eta_q}$. We have

14.16. Proposition. *Let $p, q \in \mathbb{N}_+$ & $q \leqslant p$. Then*
(*a*) $F \in \mathcal{P}_{1,\eta_p \times \eta_q}$ *iff* $F \in L_2(\mathbb{R}^{p+q})$ &

$$\sum_{r=1}^{q} \sum_{\pi \in \mathring{\Pi}_r^{p+q}} \int_{\mathbb{R}^{p+q-2r}} \left[\int_{\mathbb{R}^r} |F_\pi^{p+q}(\tau, h)| \ell_r(\mathrm{d}\tau) \right]^2 \ell_{p+q-2r}(\mathrm{d}h) < \infty.$$

(*b*) $\forall F \in \mathcal{P}_{1,\eta_p \times \eta_q}$,

$$\mathbb{E}_{\eta_p \times \eta_q}(F) = \mathbb{E}_{\eta_{p+q}}(F) + \sum_{r=1}^{q} \sum_{\pi \in \mathring{\Pi}_r^{p+q}} \int_{\mathbb{R}^{p+q-2r}} \left[\int_{\mathbb{R}^r} F_\pi^{p+q}(\tau, h) \ell_r(\mathrm{d}\tau) \right] \eta_{p+q-2r}(\mathrm{d}h).$$

Proof. By 11.13, we have

(1) $$\eta_p \times \eta_q = \eta_{p+q} + \rho,$$

where

$$\forall D \in \mathcal{D}_p, \qquad \rho(D) := \zeta_{p+q}(D \backslash J_1^{p+q}) \in \mathcal{S}_{\zeta_{p+q}} \perp \mathcal{S}_{\eta_{p+q}}.$$

It follows from A.31(*b*) that

$$F \in \mathcal{P}_{1,\eta_p \times \eta_q} \iff F \in \mathcal{P}_{1,\eta_{p+q}} \quad \& \quad F \in \mathcal{P}_{1,\rho}$$
$$\iff F \in L_2(\mathbb{R}^{p+q}) \quad \& \quad F \in \mathcal{P}_{1,\rho},$$

i.e. by 14.15(*a*),

$$F \in \mathcal{P}_{1,\eta_p \times \eta_q} \iff F \in L_2(\mathbb{R}^{p+q}) \ \&$$
$$\sum_{r=1}^{q} \sum_{\pi \in \mathring{\Pi}_r^{p+q}} \int_{\mathbb{R}^{p+q-2r}} \left[\int_{\mathbb{R}^r} |F_\pi^{p+q}(\tau, h)| \ell_r(\mathrm{d}\tau) \right]^2 \ell_{p+q-2r}(\mathrm{d}h) < \infty.$$

Thus (*a*).

(*b*) Next, for $F \in \mathcal{P}_{1,\eta_{p+q}}$,

$$\mathbb{E}_{\eta_p \times \eta_q}(F) = \mathbb{E}_{\eta_{p+q}}(F) + \mathbb{E}_\rho(F), \qquad \text{by (1)}.$$

Substituting the value of $\mathbb{E}_\rho(F)$, given by 14.15(b), we get (b). ■

An especially useful application of proposition 14.16 is to the case where F is a tensor product of $p+q$ functions, each in $L_2(\mathbb{R})$. We have:

14.17. Corollary. *Let* $p, q \in \mathbb{N}_+$, $q \leqslant p$ & $\phi_1, \ldots, \phi_{p+q} \in L_2(\mathbb{R})$. *Then*
(a)

$$\mathop{\times}_{k=1}^{p+q} \phi_k \in \mathcal{P}_{1,\eta_p \times \eta_q};$$

(b)

$$\mathbb{E}_{\eta_p}\left(\mathop{\times}_{i=1}^{p} \phi_i\right)\mathbb{E}_{\eta_q}\left(\mathop{\times}_{j=p+1}^{p+q} \phi_j\right)$$
$$= \mathbb{E}_{\eta_{p+q}}\left(\mathop{\times}_{k=1}^{p+q} \phi_k\right) + \sum_{r=1}^{q} \sum_{\pi \in \mathring{\Pi}_r^{p+q}} \left[\left\{\prod_{\Delta \in \pi} (\phi_{\min \Delta}, \phi_{\max \Delta})\right\} \cdot \mathbb{E}_{\eta_{p+q-2r}}\left(\mathop{\times}_{m \in M'_\pi} \phi_m\right)\right].$$

Proof. (a) It is easier to prove (a) directly rather than to deduce it from 14.16(a). Since by proposition 11.13,

$$(1) \qquad \eta_p \times \eta_q = \eta_{p+q} + \rho, \qquad \rho(\cdot) := \xi_{p+q}(\cdot \backslash J_1^{p+q}),$$

we need only show, cf. A.30(b), that

$$(\mathrm{I}) \qquad \mathop{\times}_{k=1}^{p+q} \phi_k \in \mathcal{P}_{1,\eta_{p+q}} \quad \& \quad \mathop{\times}_{k=1}^{p+q} \phi_k \in \mathcal{P}_{1,\rho}.$$

Proof of (I). Since each $\phi_k \in L_2(\mathbb{R})$, the classical Tonelli theorem shows that

$$\mathop{\times}_{k=1}^{p+q} \phi_k \in L_2(\mathbb{R}^{p+q}) = \mathcal{P}_{1,\eta_{p+q}}.$$

Next, since each $\phi_k = L_2(\mathbb{R}) = \mathcal{P}_{\xi_1}$, therefore by corollary 14.11,

$$(2) \qquad F := \mathop{\times}_{k=1}^{p+q} \phi_k \in \mathcal{P}_{1,\xi_{p+q}}, \quad \text{i.e.} \quad |F|_{1,\xi_{p+q}} < \infty.$$

But by (1) and 9.13(h), $\forall x' \in (\mathcal{L}_2)'$ & $\forall A \in \mathcal{B}_{p+q}$,

$$|x \circ \rho|(A) = |x' \circ \zeta_{p+q}|(A \backslash J_1^{p+q}) \leqslant |x' \circ \zeta_{p+q}|(A) \leqslant |x' \circ \xi_{p+q}|(A).$$

It follows that $|F|_{1,\rho} \leqslant |F|_{1,\xi_{p+q}} < \infty$, by (2), i.e. $F \in \mathcal{P}_{1,\rho}$. Thus (I) is established and (a) proved.

(b) Write $\varPhi := \times_{k=1}^{p} \phi_i$, $\varPsi := \times_{j=p+1}^{p+q} \phi_j$. Then by the classical Tonelli theorem,

$$\varPhi \in L_2(\mathbb{R}^p) = \mathcal{P}_{1,\eta_p} \quad \& \quad \varPsi \in L_2(\mathbb{R}^q) = \mathcal{P}_{1,\eta_q}.$$

Hence by (a), the Fubini proposition 14.1 and proposition 14.16(b),

$$\mathrm{LHS}(b) := \mathbb{E}_{\eta_p}(\varPhi) \cdot \mathbb{E}_{\eta_q}(\varPsi) = \mathbb{E}_{\eta_p \times \eta_q}(\varPhi \times \varPsi) = \mathbb{E}_{\eta_p \times \eta_q}(F)$$
$$(3) \qquad = \mathbb{E}_{\eta_{p+q}}(F) + \sum_{r=1}^{q} \sum_{\pi \in \mathring{\Pi}_r^{p+q}} \int_{\mathbb{R}^{p+q-2r}} \left[\int_{\mathbb{R}^r} F_\pi^{p+q}(\tau, h) \ell_r(\mathrm{d}\tau)\right] \eta_{p+q-2r}(\mathrm{d}h).$$

But $F := \times_{k=1}^{p+q}\phi_k$, and hence by 12.20($b$), $\forall \pi \in \Pi_r^{p+q}$ & $\forall h \in \mathbb{R}^{p+q-2r}$,

$$\int_{\mathbb{R}^r} F_\pi^{p+q}(\tau, h)\ell_r\,(\mathrm{d}\tau) = \prod_{\Delta\in\pi}(\phi_{\min\Delta}, \phi_{\max\Delta}) \cdot \Big(\bigtimes_{m\in M'_\pi}\phi_m\Big)(h).$$

Substituting from this on the RHS of (3), and observing that $\prod_{\Delta\in\pi}(\phi_{\min\Delta}, \phi_{\max\Delta})$ is independent of h, we get the equality in (b). ■

A simple but important consequence of the last corollary pertains to the tensor product of two tensor products, such that the component functions of one are orthogonal to those of the other. This result, which is a useful lemma, reads as follows:

14.18. Lemma. *Let* (i) $p, q \in \mathbb{N}_+$ & $q \leqslant p$, (ii) $f_1, \ldots, f_p$ & $g_1, \ldots, g_q \in L_2(\mathbb{R})$ & $\forall i \in [1, p]$ & $\forall j \in [1, q]$, $f_i \perp g_j$. *Then*

$$\mathbb{E}_{\eta_{p+q}}\Big(\bigtimes_{i=1}^{p} f_i \times \bigtimes_{j=1}^{q} g_j\Big) = \mathbb{E}_{\eta_p}\Big(\bigtimes_{i=1}^{p} f_i\Big) \cdot \mathbb{E}_{\eta_q}\Big(\bigtimes_{j=1}^{q} g_j\Big).$$

Proof. Let

$$\forall k \in [1, p+q], \qquad \phi_k = \begin{cases} f_k & \text{if } k \in [1,p], \\ g_{k-p} & \text{if } k \in [p+1, p+q]. \end{cases}$$

Then $\phi_1, \ldots, \phi_{p+q} \in L_2(\mathbb{R})$ and hence $\mathbb{E}_{\eta_p}(\times_{k=1}^{p}\phi_k) \cdot \mathbb{E}_{\eta_q}(\times_{k=p+q}^{p+q}\phi_k)$ is given by the formula in 14.17(b).

Now let $r \in [1, q]$ and $\pi \in \mathring{\Pi}_r^{p+q}$. Then $\forall \Delta = \{i, j\} \in \pi$, $1 \leqslant i \leqslant p < j \leqslant p+q$, and therefore by (i), $\phi_i := f_i \perp g_{j-p} =: \phi_j$, i.e. $(\phi_{\min\Delta}, \phi_{\max\Delta}) = 0$. It follows that the second term on the RHS of 14.17(b) vanishes, and therefore

$$(1) \qquad \mathbb{E}_{\eta_p}\Big(\bigtimes_{i=1}^{p}\phi_i\Big) \cdot \mathbb{E}_{\eta_q}\Big(\bigtimes_{j=p+1}^{p+q}\phi_j\Big) = \mathbb{E}_{\eta_{p+q}} = \Big(\bigtimes_{k=1}^{p+q}\phi_k\Big).$$

In (1), $\phi_i = f_i$ and $\phi_k = g_{k-p}$. Hence, writing $j := k - p$, (1) reduces to the desired equality. ■

Since the functions $g_1, \ldots, g_q$ in lemma 14.18, apart from being in $L_2(\mathbb{R})$ and being orthogonal to the f_k are arbitrary, we may consider the special case in which they break up into two groups that are themselves orthogonal. Then by the last lemma, $\mathbb{E}_{\eta_q}(\times_{j=1}^{q} g_j)$ will itself factor into a product of two integrals, say

$$\mathbb{E}_{\eta_{q_1}}\Big(\bigtimes_{k=1}^{q_1} g_k\Big) \cdot \mathbb{E}_{\eta_{q_2}}\Big(\bigtimes_{k=q_1+1}^{q_2} g_j\Big) \quad \text{where} \quad q_1 + q_2 = q.$$

The substitution of this on the RHS of 14.17, will yield a factorization of the LHS of 14.17 into three fractors. This process can be repeated any finite number of times. We thus arrive at the following important theorem:

14.19. Theorem. (Tensor product of orthogonal blocks) *Let* (i) $n \in \mathbb{N}_+$,
(ii) $\forall i \in [1, n]$, $p_i \in \mathbb{N}_+$ & $\forall j \in [1, p_i]$, $f_{ij} \in L_2(\mathbb{R})$,
(iii) $\forall i, i' \in [1, n] \ni i \neq i'$, $\{f_{i,1}, \ldots, f_{i,p_i}\} \perp \{f_{i',1}, \ldots, f_{i',p_{i'}}\}$.

Then

$$\mathbb{E}_{\eta_{p_1+\cdots+p_n}}\left(\bigtimes_{i=1}^{n}\bigtimes_{j=1}^{p_i} f_{i,j}\right) = \prod_{i=1}^{n}\mathbb{E}_{\eta_{p_i}}\left(\bigtimes_{j=1}^{p_i} f_{i,j}\right).$$

Proof. By 14.18 the equality holds for $n = 2$. Assume that it holds for n and define ϕ_k for $\forall k \in [1, p_1 + \cdots + p_n]$ as follows

$$\begin{aligned} \text{for}\quad & k \in [1, p_1], \quad \phi_k = f_{1,k}, \\ & k \in [p_1 + 1, p_1 + p_2], \quad \phi_k = f_{2,k-p_1}, \\ & k \in [p_1 + p_2 + 1, p_1 + p_2 + p_3], \quad \phi_k = f_{3,k-(p_1+p_2)}, \end{aligned}$$

and so on, and let

(1) $$\Phi := \bigtimes_{k=1}^{p_1+\cdots+p_n} \phi_k \quad \& \quad \Psi := \bigtimes_{j=1}^{p_{n+1}} f_{n+1,j}.$$

Then

$$\Phi = \bigtimes_{i=1}^{n}\bigtimes_{j=1}^{p_i} f_{i,j} \quad \& \quad \bigtimes_{k=1}^{p_1+\cdots+p_{n+1}} \phi_k = \Phi \times \Psi.$$

Hence

$$\begin{aligned} \mathbb{E}_{\eta_{p_1+\cdots+p_{n+1}}}\left(\bigtimes_{k=1}^{p_1+\cdots+p_{n+1}} \phi_k\right) &= \mathbb{E}_{\eta_{(p_1+\cdots+p_n)+p_{n+1}}}(\Phi \times \Psi) \\ &= \mathbb{E}_{\eta_{p_1+\cdots+p_n}}(\Phi) \cdot \mathbb{E}_{\eta_{p_{n+1}}}(\Psi) \quad \text{by 14.18, as } \Phi \perp \Psi \\ &= \mathbb{E}_{\eta_{p_1+\cdots+p_n}}\left(\bigtimes_{i=1}^{n}\bigtimes_{j=1}^{p_i} f_{i,j}\right) \cdot \mathbb{E}_{\eta_{p+1}}\left(\bigtimes_{j=1}^{p_{n+1}} f_{n+1,j}\right) \\ &= \prod_{i=1}^{n}\mathbb{E}_{\eta_{p_i}}\left(\bigtimes_{j=1}^{p_i} f_{i,j}\right) \cdot \mathbb{E}_{\eta_{p+1}}\left(\bigtimes_{j=1}^{p_{n+1}} f_{n+1,j}\right) \qquad (2) \\ &= \prod_{i=1}^{n+1}\mathbb{E}_{\eta_{p_i}}\left(\bigtimes_{j=1}^{p_i} f_{i,j}\right), \end{aligned}$$

(2) by the inductive assumption. Thus the equality holds for $n + 1$. ∎

From this powerful theorem, the following important theorem of K. Ito follows at once:

14.20. Theorem. (Ito 1951) *Let* $n \in \mathbb{N}_+$ & $f_1, \ldots, f_n \in L_2(\mathbb{R})$ *be mutually orthogonal. Then* $\forall p_1, \ldots, p_n \in \mathbb{N}_+$,

$$\mathbb{E}_{\eta_{p_1+\cdots+p_n}}\left\{\bigtimes_{i=1}^{n} f_i^{\times p_i}\right\} = \prod_{i=1}^{n}\mathbb{E}_{\eta_{p_i}}\{f_i^{\times p_i}\}.$$

Proof. Let $\forall i \in [1, n]$ and $\forall j \in [1, p]$, $f_{i,j} := f_i$. Then the $f_{i,j}$ satisfy the premises of 14.19, and of course each $\bigtimes_{j=1}^{p_i} f_{i,j} = f_i^{\times p_i}$. Hence the equality in 14.19 reduces to the one enunciated. ∎

The set-theoretic analogue of Ito's theorem, also very important, emerges as an immediate corollary:

14.21. Corollary. (Ito's theorem for sets) *Let* $n \in \mathbb{N}_+$ & $A_1, \ldots, A_n \in \mathcal{D}_1$ *be mutually disjoint. Then* $\forall p_1, \ldots, p_n \in \mathbb{N}_+$,

$$\eta_{p_1+\cdots+p_n}(A_1^{p_1} \times \cdots \times A_n^{p_n}) = \eta_{p_1}(A_1^{p_1}) \cdots \eta_{p_n}(A_n^{p_n}).$$

Proof. Let $\forall i \in [1, n]$, $f_i = \chi_{A_i}$. Then $f_i \in L_2(\mathbb{R})$, and since $A_i \| A_j$,therefore $f_i \perp f_j$. Hence the equality of theorem 14.20 holds. But this equality reduces to that in 14.21, since for $\forall i, j \in [1, n]$,

$$f_i^{\times p_i} = (\chi_{A_i})^{\times p_i} = \chi_{A_i^p} \quad \& \quad \bigtimes_{i=1}^{n} f_i^{\times p_i} = \bigtimes_{i=1}^{n} \chi_{A_i^{p_i}} = \chi_{A_i^{p_i} \times \cdots \times A_n^{p_n}}.$$

■

15. The inversion formulae

Let $p \in \mathbb{N}_+$ and $D \in \mathcal{D}_p$. Our objective is to show that the inversion of the formula in 11.10, namely,

(1) $$\xi_p(D) = \sum_{k=0}^{[p/2]} \int_{\mathbb{R}^{p-2k}} \gamma_k^p(D, h)\eta_{p-2k}(\mathrm{d}h),$$

takes the form

(2) $$\eta_p(D) = \sum_{k=0}^{[p/2]} (-1)^k \int_{\mathbb{R}^{p-2k}} \gamma_k^p(D, h)\xi_{p-2k}(\mathrm{d}h),$$

and that the inversion of the integral formula in 13.12, namely,

(1′) $$\mathbb{E}_{\xi_p}(f) = \sum_{k=0}^{[p/2]} \mathbb{E}_{\eta_{p-2k}}(f_k^p),$$

takes the form

(2′) $$\mathbb{E}_{\eta_p}(f) = \sum_{k=0}^{[p/2]} (-1)^k \mathbb{E}_{\xi_{p-2k}}(f_k^p).$$

The formula (2′) was worked out in essence for the cases $p = 2, 3$ by Wiener (1958, pp. 28–36).

Since $\gamma_k^p(D, \cdot)$ is a bounded measurable function on $\mathbb{R}^{p-2k}$ with support in $\mathcal{D}_{p-2k}$, cf. 4.16(*d*), it follows from (A.19) that $\gamma_k^p(D, \cdot) \in \mathcal{P}_{1,\xi_{p-2k}}$, and the integrals in (2) exist. Thus the expansion (2) is meaningful. It may be viewed as a vectorial extension of the standard expansion of the pth Hermite polynomial in the Kakutani format:

(3) $$H_p(u, \sigma) = \sum_{k=0}^{[p/2]} (-1)^k \binom{p}{2k} \alpha_{2k} \sigma^k u^{p-2k}, \quad u \in \mathbb{R} \quad \& \quad \sigma \in \mathbb{R}_+, \quad \text{cf. 16.2.}$$

Indeed the RHSs of (2) and (3) match, upon taking $D = A^p$, $u = \xi_1(A)$ and $\sigma = |\xi_1(A)|^2$, cf. 16.3(*c*) below.

Since, cf. 9.6, $\eta_p(D) = \mathrm{Proj}(\xi_p(D) | \mathcal{S}_{\xi_{p-2}}^{\perp})$, the formula (2) may also be viewed

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as the outcome of a Gram–Schmidt process applied to the sequence of measures $\{\xi_0, \xi_2, \xi_4, \ldots\}$ or $\{\xi_1, \xi_3, \xi_5, \ldots\}$ in which integral signs rather than the usual sigmas appear. Such a process can be carried out for small p, such as 2, 3, 4, and for sets D, which are intervals, by adopting Wiener's approach to find the functional counterparts of $\eta_n(\cdot)$, namely $G_n\{K(\cdot)\}$ (cf. Wiener 1958, pp. 28–36).

The inversion (2) is, however, very difficult to prove in full generality. It is a fundamental fact that $\forall P \in \mathcal{P}_p$, $\eta_p(P)$ has to be a linear combination of the ξ_{p-2k} measures of the $p-2k$ dimensional faces of P, for $k \in [0, [p/2]]$; we have:

15.1. Proposition. *Let* $p \in \mathbb{N}_+$ *&* $P \in \mathcal{P}_p$*. Then* $\forall k \in [0, [p/2]]$, $\forall M \subseteq [1, p] \ni \#M = 2k$, $\exists c(P, M) \in \mathbb{R}$ *such that*

$$\eta_p(P) = \sum_{k=0}^{[p/2]} \left\{ \sum_{\substack{M \subseteq [1,p] \\ \#M = 2k}} c(P, M)\xi_{p-2k}(P_{M'}) \right\}, \quad \textit{where} \quad M' := [1, p] \backslash M.$$

This result, in the form

$$\eta_p(P) \in \langle \xi_{p-2k}(P_{M'}) : k \in [0, [1, p/2]], M \subseteq [1, p] \ \& \ \#M = 2k \rangle,$$

where $\langle S \rangle$ stands for the linear manifold spanned by S, can be proved by induction. It is of small use unless accompanied by a statement of the coefficients $c(P, M)$. Inspection with small p clearly indicates that $\forall M \subseteq [1, p]$ with $\#M = 2k$, $c(P, M) = (-1)^k \sum_{\pi \in \Pi_k^p} a_\pi^p(P)$, where $a_\pi^p(P)$ are the very coefficients defined in (3.11). One is thus led to conjecture (correctly, it turns out) that in general,

$$(4) \qquad \forall p \in \mathbb{N}_+ \ \& \ \forall P \in \mathcal{P}_p, \quad \eta_p(P) = \sum_{k=0}^{[p/2]} (-1)^k \left\{ \sum_{\pi \in \Pi_k^p} a_\pi^p(P)\xi_{p-2k}(P_{M'_\pi}) \right\}.$$

Attempts at a direct proof of (4) get bogged down in a combinatoric quagmire, however. We shall therefore follow an alternative, more general, approach suggested by the following considerations.

Our task is to invert the infinite triangular system of integral equations, the pth equation of which is (1). Thus our problem falls in the arena of so-called *Möbius inversion* (cf. Gian-Carlo Rota 1964). The validity of the equation (2) for small p gives us the valuable hint that the solution has the same integrands but with alternating signs. The task is analogous to that of inverting an infinite triangular matrix $[a_{ij}]$ and showing that $[a_{ij}]^{-1} = [(-1)^{i+j} a_{ij}]$. For the last equality to prevail, the a_{ij} must satisfy an infinite number of recurrence relations such as

$$2a_{42} = a_{43} \cdot a_{32}, \quad 2a_{51} = a_{52} \cdot a_{21} - a_{53} \cdot a_{31} + a_{54} \cdot a_{41}, \ldots.$$

In our problem, inspection with small p clearly suggests that the recurrence relations that ought to govern the canonical coefficients $\gamma_k^p(\cdot, \cdot)$ are:

$$(5) \quad \binom{k}{j} \gamma_k^p(D, h) = \int_{\mathbb{R}^{p-2j}} \gamma_j^p(D, t)\gamma_{k-j}^{p-2j}(\mathrm{d}t, h), \qquad D \in \mathcal{D}_p, \qquad h \in \mathbb{R}^{p-2k},$$

for $0 \leqslant j \leqslant k \leqslant [p/2]$. With their aid it becomes possible to prove the general formula (2). The formula (4) then follows as a special case of (2) and this in turn establishes the proposition 15.1.

We proceed to establish (5). Almost all the difficult combinatorial questions that arise in the proof can be reduced to questions about a certain division operation $A|B$ defined for sets of positive integers when $A \subseteq B$, and about the relationships between the classes of partitions $\Pi_{B\setminus A}$ and $\Pi_{(B\setminus A)|B}$, when $B\setminus A$ has even cardinality. To avoid a digression, this purely combinatorial study is relegated to Appendix C. This the reader should now consult, since extensive use is made of it.

The following fundamental recurrence relation governing the canonical coefficients $\lambda^p_\pi(\cdot, h)$ of 4.13 is an essential stepping stone on the way to the recurrence (5) governing the coefficients $\gamma^p_k(\cdot, h)$.

15.2. Lemma. (Recurrence equation for $\lambda^p_\pi(D,h)$) *Let*
(i) $p \in \mathbb{N}_+$ & $1 \leqslant j \leqslant k \leqslant [p/2]$,
(ii) $\pi_1 \in \Pi^p_j$, $\pi_2 \in \Pi^p_{k-j}$ *be such that* $M_{\pi_2} \subseteq [1,p]\setminus M_{\pi_1} =: M'_{\pi_1}$,
(iii) $\bar\pi_2 :=$ *the canonical* $M_{\pi_2}|M'_{\pi_1}$ *associate of* π_2 *(cf. C.7).*
Then (a) $\pi_1 \cup \pi_2 \in \Pi^p_k$, $\bar\pi_2 \in \Pi^{p-2j}_{k-j}$ & $M_{\bar\pi_2} = M_{\pi_2}|M'_{\pi_1}$;
(b) $\forall D \in \mathcal{D}_p$ & $\forall h \in \mathbb{R}^{p-2k}$,

$$\lambda^p_{\pi_1\cup\pi_2}(D,h) = \int_{\mathbb{R}^{p-2j}} \lambda^p_{\pi_1}(D,t)\lambda^{p-2j}_{\bar\pi_2}(\mathrm{d}t,h).$$

Proof. (a) Obviously by (ii), $\pi_1 \cup \pi_2 \in \Pi^p_k$. Next by C.2, $M_{\pi_2}|M'_{\pi_1} \subseteq [1, \#(M'_{\pi_1})] = [1, p-2j]$. Hence, cf. C.7 and Note, $\bar\pi_2 \in \Pi^{p-2j}_{k-j}$ & $M_{\bar\pi_2} = M_{\pi_2}|M'_{\pi_1}$. Thus (a).

(b) Grant momentarily that

(I) the equality in (b) holds for all $D \in \mathcal{P}_p$,

and let for π_1, π_2 as in (ii) and for fixed $h \in \mathbb{R}^{p-2k}$,

$$\mu(D) := \lambda^p_{\pi_1\cup\pi_2}(D,h) \quad \& \quad \nu(D) := \int_{\mathbb{R}^{p-2j}} \lambda^p_{\pi_1}(D,t)\lambda^{p-2j}_{\bar\pi_2}(\mathrm{d}t,h), \qquad D \in \mathcal{D}_p.$$

Then by 4.15(a), $\mu \in \mathrm{CA}(\mathcal{D}_p, \mathbb{R}_{0+})$ & $\lambda^{p-2j}_{\bar\pi_2}(\cdot,h) \in \mathrm{CA}(\mathcal{D}_{p-2j}, \mathbb{R}_{0+})$. Therefore by B.1$(a)$, $\nu \in \mathrm{CA}(\mathcal{D}_p, \mathbb{R}_{0+})$. Moreover by (I), $\mu = \nu$ on $\mathcal{P}_p$. Hence by the identity principle A.8, $\mu = \nu$ on δ-ring $(\mathcal{P}_p) = \mathcal{D}_p$. Thus it only remains to justify (I).

Proof of (I). Let $P \in \mathcal{P}_p$ and $h \in \mathbb{R}^{p-2k}$. Since $\pi_1 \in \Pi^p_j$, therefore by (4.19),

$$\forall t \in \mathbb{R}^{p-2j}, \qquad \lambda^p_{\pi_1}(P,t) = \ell_j\{P(\pi_1)\}\chi_{P_{M'_{\pi_1}}}(t). \tag{1}$$

Now $\#M'_{\pi_1} = p-2j$. Hence

$$P_{M'_{\pi_1}} := \bigtimes_{i\in M'_{\pi_1}} P^i \in \mathcal{P}_{p-2j}. \tag{2}$$

Next, since by (a), $\bar\pi_2 \in \Pi^{p-2j}_{k-j}$ & $\#M_{\bar\pi_2} = 2\#\bar\pi_2 = 2(k-j)$, therefore

$$\#\{[1,p-2j]\setminus M_{\bar\pi_2}\} = (p-2j) - 2(k-j) = p-2k. \tag{3}$$

By (2), (3) and 4.13, the symbol $\lambda^{p-2j}_{\bar\pi_2}(P_{M'_{\pi_2}}, h)$ is well defined. Hence by (1),

$$\begin{aligned}\int_{\mathbb{R}^{p-2j}} \lambda^p_{\pi_1}(P,t)\lambda^{p-2j}_{\bar\pi_2}(\mathrm{d}t,h) &= \int_{\mathbb{R}^{p-2j}} \ell_j\{P(\pi_1)\}\chi_{P_{M'_{\pi_1}}}(t)\lambda^{p-2j}_{\bar\pi_2}(\mathrm{d}t,h)\\ &= \ell_j\{P(\pi_1)\}\lambda^{p-2j}_{\bar\pi_2}(P_{M'_{\pi_1}},h).\end{aligned} \tag{4}$$

Now grant momentarily that

(II) $$\lambda^{p-2j}_{\bar\pi_2}(P_{M'_{\pi_1}},h)=\ell_{k-j}[P(\pi_2)]\chi_{P_{M'_{\pi_1\cup\pi_2}}}(h).$$

Then

$$\begin{aligned}\mathrm{RHS}(4)&=\ell_j\{P(\pi_1)\}\cdot\ell_{k-j}[P(\pi_2)]\chi_{P_{M'_{\pi_1\cup\pi_2}}}(h)=\ell_k\{P(\pi_1)\times P(\pi_2)\}\chi_{P_{M'_{\pi_1\cup\pi_2}}}(h)\\&=\ell_k\{P(\pi_1\cup\pi_2)\}\chi_{P_{M'_{\pi_1\cup\pi_2}}}(h)=:\lambda^p_{\pi_1\cup\pi_2}(P,h).\end{aligned}$$

Substituting in (4) we get the desired equality. This completes the proof of (I) and establishes (b) but for the justification of (II).

Proof of (II). By (2), (3) and (1),

(5) $$\lambda^{p-2j}_{\bar\pi_2}(P_{M'_{\pi_1}},h)=\ell_{k-j}[(P_{M'_{\pi_1}})(\bar\pi_2)]\chi_{(P_{M'_{\pi_1}})_{M'_{\bar\pi_2}}}(h),$$

where by (a) and C.2(b),

$$M'_{\bar\pi_2}:=[1,p-2j]\backslash M_{\bar\pi_2}=[1,\#M'_{\pi_1}]\backslash(M_{\pi_2}|M'_{\pi_1})=(M'_{\pi_1}\backslash M_{\pi_2})|M'_{\pi_1}.$$

Now trivially

$$\begin{aligned}M'_{\pi_1}\backslash M_{\pi_2}=([1,p]\backslash M_{\pi_1})\backslash M_{\pi_2}&=[1,p]\backslash(M_{\pi_1}\cup M_{\pi_2})\\&=[1,p]\backslash M_{\pi_1\cup\pi_2}=M'_{\pi_1\cup\pi_2}.\end{aligned}$$

Thus $M'_{\bar\pi_2}=M'_{\pi_1\cup\pi_2}|M'_{\pi_1}$, and therefore by C.3,

$$(P_{M'_{\pi_1}})_{M'_{\bar\pi_2}}=(P_{M'_{\pi_1}})_{M'_{\pi_1\cup\pi_2}|M'_{\pi_1}}=P_{M'_{\pi_1\cup\pi_2}}.$$

We also note that by C.8(b), $(P_{M'_{\pi_1}})(\bar\pi_2)=P(\pi_2)$. Substituting these values in (5), we get (II). ∎

Getting the corresponding integral equation for the coefficients γ^p_k from the one just obtained for the $\lambda^p_{\pi_1\cup\pi_2}$ is a matter of complicated combinatorics, requiring an appeal to the summation lemma C.5 in Appendix C. We have:

15.3. Theorem. *Let* $p\in\mathbb{N}_+$ *and* $0\leqslant j\leqslant k\leqslant[p/2]$. *Then* $\forall D\in\mathcal{D}_p$ *and* $\forall h\in\mathbb{R}^{p-2k}$,

$$\binom{k}{j}\gamma^p_k(D,h)=\int_{\mathbb{R}^{p-2j}}\gamma^p_j(D,t)\gamma^{p-2j}_{k-j}(\mathrm{d}t,h).$$

Proof. Let $D\in\mathcal{D}_p$ and $h\in\mathbb{R}^{p-2k}$. For $j=0$, we have $\gamma^p_j(D,h)=\chi_D(h)$, cf. 4.13 Note, whence the equality follows trivially. For $j=k$, again $\gamma^{p-2j}_{k-j}(\varDelta,h)=\chi_\varDelta(h)=m_h(\varDelta)$, where m_h is the unit mass carried at h. The RHS of the equality reduces to $\gamma^p_j(D,h)$ as does the LHS. Thus the result is again trivial.

Let $1\leqslant j<k\leqslant[p/2]$. To deal with the LHS of the enunciated equality, let

$$\forall\pi\in\varPi^p_k\ \&\ \forall J\subseteq[1,k],\quad \pi_J:=\{\varDelta_i:\varDelta_i\in\pi\ \&\ i\in J\}.$$

Then with $J':=[1,k]\backslash J$, we have

(1) $$\forall\pi\in\varPi^p_k\quad\&\quad\forall J\subseteq[1,k],\quad\pi_J\cup\pi_{J'}=\pi.$$

Now define

(2) $$\varPi^p_{j,k}:=\{(\pi_J,\pi_{J'}):\exists\pi\in\varPi^p_k,\exists J\subseteq[1,k]\ \&\ \#J=j\}=\bigcup_{\pi\in\varPi^p_k}\ \bigcup_{\substack{J\subseteq[1,k]\\ \#J=j}}\{(\pi_J,\pi_{J'})\}.$$

We assert that

$$\text{(I)}\qquad \forall D \in \mathcal{D}_p \quad \& \quad \forall h \in \mathbb{R}^{p-2k}, \qquad \sum_{(\pi_1,\pi_2)\in\Pi^p_{j,k}} \lambda^p_{\pi_1\cup\pi_2}(D,h) = \binom{k}{j}\gamma^p_k(D,h).$$

Proof of (I). Let $D \in \mathcal{D}_p$ & $h \in \mathbb{R}^{p-2k}$. Then by (2),

$$\begin{aligned}\text{LHS(I)} &= \sum_{\pi\in\Pi^p_k}\sum_{\substack{J\subseteq[1,k]\\ \#J=j}} \lambda^p_{\pi_J\cup\pi_{J'}}(D,h)\\ &= \sum_{\pi\in\Pi^p_k}\sum_{\substack{J\subseteq[1,k]\\ \#J=j}} \lambda^p_{\pi}(D,h) = \sum_{\substack{J\subseteq[1,k]\\ \#J=j}}\sum_{\pi\in\Pi^p_k} \lambda^p_{\pi}(D,h) \qquad \text{by (1)},\\ &= \sum_{\substack{J\subseteq[1,k]\\ \#J=j}} \gamma^p_k(D,h) = \binom{k}{j}\gamma^p_k(D,h).\end{aligned}$$

Thus (I).

To deal with the RHS of the enunciated equality, we assert first that

$$\text{(II)}\quad \begin{cases} \text{given } L\subseteq[1,p],\ \#L=2j\ \&\ L' := [1,p]\backslash L,\ \forall\Delta\in\mathcal{D}_{k-j}\ \&\ \forall h\in\mathbb{R}^{p-2k},\\ \qquad \gamma^{p-2j}_{k-j}(\Delta,h) = \displaystyle\sum_{\substack{L\subseteq M\subseteq[1,p]\\ \#M=2k}}\ \sum_{\pi\in\Pi_{M\backslash L}} \lambda^{p-2j}_{\bar\pi}(\Delta,h),\\ \text{where } \bar\pi \text{ is the } M_\pi|L' \text{ associate of } \pi\in\Pi_{M\backslash L}, \text{ cf. C.7.}\end{cases}$$

Proof of (II). Let $\Delta \in \mathcal{D}_{k-j}$ and $h \in \mathbb{R}^{p-2k}$. Since $p-2k = (p-2j) - 2(k-j)$, therefore

$$\begin{aligned}\gamma^{p-2j}_{k-j}(\Delta,h) &= \sum_{\pi\in\Pi^{p-2j}_{k-j}} \lambda^{p-2j}_\pi(\Delta,h) = \sum_{\substack{N\subseteq[1,p-2j]\\ \#N=2(k-j)}}\sum_{\pi\in\Pi_N} \lambda^{p-2j}_\pi(\Delta,h) \quad \text{by 1.16}(d),\\ &= \sum_{\substack{N\subseteq[1,p-2j]\\ \#N=2(k-j)}} g(N), \quad \text{where} \quad g(N) := \sum_{\pi\in\Pi_N}\lambda^{p-2j}_\pi(\Delta,h),\\ &= \sum_{\substack{L\subseteq M\subseteq[1,p]\\ \#M=2k}} g\{(M\backslash L)|L'\} \qquad\qquad \text{by lemma C.5},\\ &= \sum_{\substack{L\subseteq M\subseteq[1,p]\\ \#M=2k}}\ \sum_{\pi\in\Pi_{(M\backslash L)|L'}} \lambda^{p-2j}_\pi(\Delta,h).\end{aligned}$$

But by C.8(a), $\Pi_{(M\backslash L)|L'} = \{\bar\pi : \pi\in\Pi_{M\backslash L}\}$. Hence

$$\gamma^{p-2j}_{k-j}(\Delta,h) = \sum_{\substack{L\subseteq M\subseteq[1,p]\\ \#M=2k}}\ \sum_{\pi\in\Pi_{M\backslash L}} \lambda^{p-2k}_{\bar\pi}(\Delta,h).$$

Thus (II).

Now let $D\in\mathcal{D}_p$, $t\in\mathbb{R}^{p-2j}$, $\Delta\in\mathcal{D}_{p-2j}$ & $h\in\mathbb{R}^{p-2k}$. Then by 4.13 and (II), in which we take $L = M_{\pi_1}$, and replace M by N for notational clarity,

$$\gamma_j^p(D,t)\gamma_{k-j}^{p-2j}(\Delta,h) = \sum_{\pi_1\in\Pi_j^p}\lambda_{\pi_1}^p(D,t)\sum_{\substack{M_{\pi_1}\subseteq N\subseteq[1,p]\\ \#N=2k}}\sum_{\pi_2\in\Pi_{N\setminus M_{\pi_1}}}\lambda_{\bar{\pi}_2}^{p-2j}(\Delta,h)$$
$$= \sum_{\pi_1\in\Pi_j^p}\sum_{\substack{M_{\pi_1}\subseteq N\subseteq[1,p]\\ \#N=2k}}\sum_{\pi_2\in\Pi_{N\setminus M_{\pi_1}}}\lambda_{\pi_1}^p(D,t)\lambda_{\bar{\pi}_2}^{p-2j}(\Delta,h).$$

From this it follows that

$$\int_{\mathbb{R}^{p-2j}}\gamma_j^p(D,t)\gamma_{k-j}^{p-2j}\,(\mathrm{d}t,h) = \sum_{\pi_1\in\Pi_j^p}\sum_{\substack{M_{\pi_1}\subseteq N\subseteq[1,p]\\ \#N=2k}}\sum_{\pi_2\in\Pi_{N\setminus M_{\pi_1}}}\int_{\mathbb{R}^{p-2j}}\lambda_{\pi_1}^p(D,t)\lambda_{\bar{\pi}_2}^{p-2j}\,(\mathrm{d}t,h)$$

(3)
$$= \sum_{\pi_1\in\Pi_j^p}\sum_{\substack{M_{\pi_1}\subseteq N\subseteq[1,p]\\ \#N=2k}}\sum_{\pi_2\in\Pi_{N\setminus M_{\pi_1}}}\lambda_{\pi_1\cup\pi_2}^p(D,h),$$

by lemma 15.2, which is appliable since $\pi_2 \in \Pi_{N\setminus M_{\pi_1}}$, and therefore $\pi_2 \in \Pi_{k-j}$ & $M_{\pi_2}\| M_{\pi_1}$.

Comparing the LHS(I) and the RHS(3), we see that to complete the proof we need only show that for any numerical function f on $\Pi_{j,k}^p$,

$$\sum_{(\pi_1,\pi_2)\in\Pi_{j,k}^p} f(\pi_1,\pi_2) = \sum_{\pi_1\in\Pi_j^p}\sum_{\substack{M_{\pi_1}\subseteq N\subseteq[1,p]\\ \#N=2k}}\sum_{\pi_2\in\Pi_{N\setminus M_{\pi_1}}} f(\pi_1,\pi_2),$$

which would of course follow, provided that

(III)
$$\Pi_{j,k}^p = \{(\pi_1,\pi_2) : \pi_1\in\Pi_j^p \ \&\ \exists N \ni M_{\pi_1}\subseteq N\subseteq[1,p] \ \&\ \#N = 2k \ \&\ \pi_2\in\Pi_{N\setminus M_{\pi_1}}\}.$$

We shall complete the proof by establishing (III).

Proof of (III). Let $(\pi_J,\pi_{J'})\in\Pi_{j,k}^p$. Then by (2), $\pi\in\Pi_k^p$, $J\subseteq[1,k]$ & $\#J = j$. Now define

$$N := M_\pi, \qquad \pi_1 := \pi_J \quad \& \quad \pi_2 := \pi_{J'}.$$

Then since $\#J = j$, therefore $\pi_1 \in \Pi_j^p$. Since by (1) $\pi = \pi_J\cup\pi_{J'}$, therefore $M_{\pi_1}\subseteq M_\pi = N \subseteq [1,k]$ and $M_{\pi_2} = M_\pi\setminus M_{\pi_1} = N\setminus M_{\pi_1}$, i.e. $\pi\in\Pi_{N\setminus M_{\pi_1}}$. Thus $(\pi_1,\pi_2)\in$ RHS(III).

Next, let $(\pi_1,\pi_2)\in$ RHS(III). Then $M_{\pi_1}\subseteq N$ & $M_{\pi_2} = N\setminus M_{\pi_1}$. Therefore $\#M_{\pi_2} = \#N - \#M_{\pi_1} = 2k-2j = 2(k-j)$ and so $\pi_2\in\Pi_{k-j}^p$. Define $\pi := \pi_1\cup\pi_2$. Then since $M_{\pi_1}\| M_{\pi_2}$, therefore $\pi_1\|\pi_2$. Hence $\#\pi = \#\pi_1 + \#\pi_2 = j+k-j = k$, i.e. $\pi\in\Pi_k^p$. Let $\pi = \{\Delta_1,\dots,\Delta_k\}$ and define $J := \{i : \Delta_i\in\pi_1\}$. Then $\#J = \#\pi_1 = j$. And

$$\pi_J := \{\Delta_i : i\in J \ \&\ \Delta_i\in\pi\} = \pi_1$$
$$\&\ \pi_{J'} := \{\Delta_i : i\in J' \ \&\ \Delta_i\in\pi\} = \{\Delta_i : \Delta_i\in\pi\setminus\pi_1\} = \pi_2.$$

Thus $(\pi_1,\pi_2) = (\pi_J,\pi_{J'})\in\Pi_{j,k}^p$. Thus (III). ∎

From theorem 15.3 we can deduce the inversion formula (2). However, it is more economical to establish the inversion (2′) first, and to get (2) from it as a special case. To establish (2′), however, it is necessary to get a connecting formula for functional marginalization (§ 12) akin to that for the canonical coefficients given in theorem 15.3. The required formula is easily revealed by reframing 15.3 in terms of indicator functions. But the proof, while considerably simpler than that of 15.3, is not entirely obvious, and as with other justifications of the heuristic rule 12.8 calls for an appeal to Appendix B.

15.4. Lemma. *Let* $p \in \mathbb{N}_+$ & $0 \leqslant j \leqslant k \leqslant [p/2]$. *Then* $\forall f \in \mathcal{P}_{1,\xi_p}$,

$$f_j^p \in \mathcal{M}_{k-j}^{p-2j} \quad \& \quad (f_j^p)_{k-j}^{p-2j} = \binom{k}{j} f_k^p, \text{ a.e. } \ell_{p-2k} \text{ on } \mathbb{R}^{p-2k}, \text{ cf. definition 12.11}(c).$$

Proof. Since $(p-2j)-2(k-j) = p-2k$, therefore by 4.16(a),

$$\forall h \in \mathbb{R}^{p-2k}, \quad \mu_h(\cdot) := \gamma_{k-j}^{p-2j}(\cdot, h) \in \mathrm{CA}(\mathcal{D}_{p-2j}, \mathbb{R}_{0+}). \tag{1}$$

Let $D \in \mathcal{D}_p$. Then by 4.16(b), $\gamma_j^p(D,\cdot)$ is measurable, bounded and boundedly supported on $\mathbb{R}^{p-2j}$. Hence

$$\begin{cases} \forall D \in \mathcal{D}_p, \quad \gamma_j^p(D,\cdot) \in L_{1,\mu_h}(\cdot) \\ \& \quad \forall s \in \mathbb{R}^{p-2j}, \quad \gamma_j^p(\cdot, s) \in \mathrm{CA}(\mathcal{D}_p, \mathbb{R}_{0+}). \end{cases} \tag{2}$$

It follows from (1) and B.9(a) that the measure $\mu_h(\cdot)$ satisfies the conditions on the measure σ imposed in (B.2) with $\mathcal{H} = \mathbb{R}$, and from (2) that the kernel $\gamma_j^p(\cdot,\cdot)$ on $\mathcal{D}_p \times \mathbb{R}^{p-2j}$ satisfies those imposed in (B.3). Hence on letting

$$\forall h \in \mathbb{R}^{p-2k} \quad \& \quad \forall D \in \mathcal{D}_p, \qquad \rho_h(D) := \int_{\mathbb{R}^{p-2j}} \gamma_j^p(D,s)\mu_h(\mathrm{d}s), \tag{3}$$

we see from theorem B.8 that $\forall h \in \mathbb{R}^{p-2k}$,

$$\begin{cases} \forall f \in L_{1,\rho_h}, \quad f_j^p(\cdot) = \int_{\mathbb{R}^p} f(t)\gamma_j^p(\mathrm{d}t,\cdot) \in L_{1,\mu_h} =: L_{1,\gamma_{k-j}^{p-2j}} \\ \& \qquad \mathbb{E}_{\rho_h}(f) = \mathbb{E}_{\mu_h}(f_j^p) := \int_{\mathbb{R}^{p-2j}} f_j^p(s)\gamma_{k-j}^{p-2j}(\mathrm{d}s,h) =: (f_j^p)_{k-j}^{p-2j}(h). \end{cases} \tag{4}$$

Next we appeal to 15.3: $\forall h \in \mathbb{R}^{p-2k}$ & $\forall D \in \mathcal{D}_p$,

$$\rho_h(D) := \int_{\mathbb{R}^{p-2j}} \gamma_j^p(D,s)\gamma_{k-j}^{p-2j}(\mathrm{d}s,h) = \binom{k}{j}\gamma_k^p(D,h). \tag{5}$$

By (5), $L_{1,\rho_h} = L_{1,\gamma_k^p(\cdot,h)}$ and $\mathbb{E}_{\rho_h} = \binom{k}{j}\mathbb{E}_{\gamma_k^p(\cdot,h)}$. Hence (4) can be rewritten

$$\begin{cases} \forall f \in L_{1,\gamma_k^p(\cdot,h)}, \quad f_j^p(\cdot) \in L_{1,\gamma_{k-j}^{p-2j}(\cdot,h)} \quad \& \\ \qquad (f_j^p)_{k-j}^{p-2j}(h) = \mathbb{E}_{\rho_h}(f) = \binom{k}{j}\mathbb{E}_{\gamma_k^p(\cdot,h)}(f). \end{cases} \tag{6}$$

Now let $f \in \mathcal{P}_{1,\xi_p}$. Then by theorem 13.12, $\forall k \in [1,[p/2]]$, $f \in \mathcal{M}_k^p$, and so by definition 12.11(b),

$$\forall k \in [1,[p/2]], \qquad H_k^p(f) = \text{a carrier of } \ell_{p-2k}. \tag{7}$$

Now fix $k \in [1,[p/2]]$, and let $h \in H_k^p(f)$. Then by 12.11(a), $f \in L_{1,\gamma_k^p(\cdot,h)}$. Hence

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by (6), $f_j^p \in L_{1,\gamma_{k-j}^{p-2j}(\cdot,h)}$, i.e. by 12.11($a$), $h \in H_{k-j}^{p-2j}(f_j^p)$. Thus $H_k^p(f) \subseteq H_{k-j}^{p-2j}(f_j^p)$. Hence by (7),

$$H_{k-j}^{p-2j}(f_j^p) = \text{a carrier of } \ell_{p-2k}, \text{ i.e. of } \ell_{(p-2j)-2(k-j)}.$$

Thus by 12.11(b),

$$f_j^p \in \mathcal{M}_{k-j}^{p-2j}. \tag{8}$$

Next, since $h \in H_k^p(f)$, therefore by 12.11(c), $\mathbb{E}_{\gamma_k^p(\cdot,h)}(f) = f_k^p(h)$, and thus the equality in (6) reduces to

$$(f_j^p)_{k-j}^{p-2j}(h) = \binom{k}{j} f_j^p(h), \qquad h \in H_k^p(f). \tag{9}$$

Since $H_k^p(f)$ is a carrier of ℓ_{p-2k}, (8) and (9) finish the proof. ∎

When we add the equations in theorem 15.4 with alternating signs, we get expressions reminiscent of those for the Euler characteristic of polytopes in $\mathbb{R}^p$:

15.5. Corollary. *Let $p \in \mathbb{N}_+$, $k \in [1, [p/2]]$. Then $\forall f \in \mathcal{P}_{1,\xi_p}$, and ℓ_{p-2k} almost all $h \in \mathbb{R}^{p-2k}$,*

(a) $$\sum_{j=0}^{k} (-1)^j (f_j^p)_{k-j}^{p-2k}(h) = 0;$$

(b) $$\sum_{j=1}^{k-1} (-1)^j (f_j^p)_{k-j}^{p-2k}(h) = \{1 + (-1)^k\} f_k^p(h).$$

Proof. (a) Let $f \in \mathcal{P}_{1,\xi_p}$. Then by 15.4, we have a.e. ℓ_{p-2k} on $\mathbb{R}^{p-2k}$,

$$\sum_{j=0}^{k} (-1)^j (f_j^p)_{k-j}^{p-2k}(h) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f_j^p = \left\{ \sum_{j=0}^{k} (-1)^j \binom{k}{j} \right\} f_j^p = 0.$$

(b) We proceed as in (a), and note that $\sum_{j=1}^{k-1} (-1)^{j+1} \binom{k}{j} = 1 + (-1)^k$. ∎

These Euler-type equations are crucial in proving the initial inversion formula (2′) to which we now turn. Note that the formula (2′) will not hold for all $f \in \mathcal{P}_{1,\eta_p}$. For, cf. 13.3, we may have $f \in \mathcal{P}_{1,\eta_p} \backslash \mathcal{P}_{1,\xi_p}$, and for such an f, the $k = 0$ term in (2′), namely, $\mathbb{E}_{\xi_p}(f_0^p) = \mathbb{E}_{\xi_p}(f)$, does not make sense. Thus the premise in 15.6 that $f \in \mathcal{P}_{1,\xi_p}$ cannot be relaxed.

15.6. Theorem. (Inversion formula for $\mathbb{E}_{\eta_p}$) *Let $p \in \mathbb{N}_+$ & $f \in \mathcal{P}_{1,\xi_p}$. Then*

(a) $$\forall j \in [0, [p/2]], \qquad f \in \mathcal{M}_j^p, \quad \& \quad f_j^p \in \mathcal{P}_{1,\xi_{p-2j}};$$

(b) $$\mathbb{E}_{\eta_p}(f) = \sum_{k=0}^{[p/2]} (-1)^k \mathbb{E}_{\xi_{p-2k}}(f_k^p).$$

Proof. (a) Let $j \in [0, [p/2]]$, and grant momentarily that

$$(\alpha) \quad |f(\cdot)| \in \mathcal{M}_j^p \qquad \& \qquad (\beta) \quad |f|_j^p \in \mathcal{P}_{1,\xi_{p-2j}}. \tag{I}$$

By (I)(α) and 12.13(b), $f \in \mathcal{M}_j^p$ and by 12.14(b), $\exists N_f \in \mathcal{N}_{\ell_{p-2k}}$ such that

$$|f_j^p(\cdot)| \leqslant |f|_j^p(\cdot) \quad \text{on} \quad \mathbb{R}^{p-2j} \backslash N_f. \tag{1}$$

By (1), (I)(β) and the domination principle (A.18), $f_j^p \in \mathcal{P}_{1,\xi_{p-2j}}$. Thus we have ($a$), once (I) is justified.

Proof of (I)(α). Write $q := p - 2j$. Then since, cf. (A.11), $F(\cdot) := |f(\cdot)| \in \mathcal{P}_{1,\xi_p}$, therefore by 13.12,

$$F \in \mathcal{M}_j^p \quad \& \quad G := F_j^p \in \mathcal{P}_{1,\eta_{p-2j}} = \mathcal{P}_{1,\eta_q}. \tag{2}$$

Thus (I)(α) is proved.

Proof of (I)(β). We have to show that $G \in \mathcal{P}_{1,\xi_q}$. Hence by 13.12, we need only show that

$$\forall i \in [1, [q/2]], \qquad G_i^q \in \mathcal{P}_{1,\eta_{q-2i}}. \tag{$*$}$$

Let $i \in [1, [q/2]]$ & $k := i + j$. Then obviously, since $q := p - 2j$,

$$i \leqslant j \leqslant k = j + i \leqslant j + [q/2] = j + [\tfrac{1}{2}p - j] \leqslant j + \tfrac{1}{2}p - j = \tfrac{1}{2}p,$$

and since $k \in \mathbb{N}_+$, we conclude that $1 \leqslant j \leqslant k \leqslant [p/2]$. Hence by definition (2) of G and the last lemma,

$$G_i^q := G_{k-j}^q = (F_j^p)_{k-j}^{p-2j} = \binom{k}{j} \cdot F_k^p. \tag{3}$$

But since $F \in \mathcal{P}_{1,\xi_p}$, therefore by 13.12, $F_k^p \in \mathcal{P}_{1,\eta_{p-2k}}$. Hence by (3), $G_i^q \in \mathcal{P}_{1,\eta_{p-2k}}$. But, $p - 2k = p - 2(i + j) = (p - 2j) - 2i = q - 2i$. Thus $G_i^q \in \mathcal{P}_{1,\eta_{q-2i}}$, i.e. we have ($*$). This proves (I)($\beta$), and finishes the proof of (a).

(b) Since by (a), $\forall k \in [0, [p/2]]$, $f_k^p \in \mathcal{P}_{1,\xi_{p-2k}}$, therefore by 13.12,

$$\begin{aligned} \mathbb{E}_{\xi_{p-2k}}(f_k^p) &= \sum_{i=0}^{[(p-2k)/2]} \mathbb{E}_{\eta_{p-2k-2i}}\{(f_k^p)_i^{p-2k}\} \\ &= \sum_{j=k}^{[p/2]} \mathbb{E}_{\eta_{p-2j}}\{(f_k^p)_{j-k}^{p-2k}\}, \qquad \text{putting } j := k + i. \end{aligned} \tag{1}$$

Hence

$$S := \sum_{k=0}^{[p/2]} (-1)^k \mathbb{E}_{\xi_{p-2k}}(f_k^p) = \sum_{k=0}^{[p/2]} (-1)^k \sum_{j=k}^{[p/2]} \mathbb{E}_{\eta_{p-2j}}\{(f_k^p)_{j-k}^{p-2k}\}.$$

Changing the order of summation and bringing out the integral operator, we get

$$S = \sum_{j=0}^{[p/2]} \mathbb{E}_{\eta_{p-2j}}\left[\sum_{k=0}^{j} (-1)^k (f_k^p)_{j-k}^{p-2k}\right] = \sum_{k=0}^{[p/2]} \mathbb{E}_{\eta_{p-2k}}\left[\sum_{j=0}^{k} (-1)^j (f_j^p)_{k-j}^{p-2j}\right], \tag{2}$$

on interchanging the dummy variables j, k. Breaking the summation into $k = 0$ and $\sum_{k=1}^{[p/2]}$, and noting that by 15.5(a), the last summation for $k \geqslant 1$ vanishes a.e. ℓ_{p-2k} on $\mathbb{R}^{p-2k}$, we see from (2) that

$$S = \mathbb{E}_{\eta_p}\{(-1)^0 (f_0^p)_0^p\} = \mathbb{E}_{\eta_p}(f), \qquad \text{cf. 12.14}(f),$$

i.e. we have (b). ■

The inversion formula (1) follows as an easy corollary of the last theorem:

15.7. Inversion theorem. *Let $p \in \mathbb{N}_+$. Then $\forall D \in \mathcal{D}_p$,*

$$\eta_p(D) = \sum_{k=0}^{[p/2]} (-1)^k \int_{\mathbb{R}^{p-2j}} \gamma_k^p(D, h) \xi_{p-2k}\,(\mathrm{d}h).$$

Proof. Let $D \in \mathcal{D}_p$. Then $\chi_D \in \mathcal{P}_{1,\xi_p}$. Hence by theorem 15.6,

$$\mathbb{E}_{\eta_p}(\chi_D) = \sum_{k=0}^{[p/2]} (-1)^k \mathbb{E}_{\xi_{p-2k}}\{(\chi_D)_k^p\}. \tag{1}$$

Since, by 12.14(d), $(\chi_D)_k^p(\cdot) = \gamma_k^p(D, \cdot)$, therefore (1) reduces to the desired formula. ■

16. Symmetric intervals, symmetric functional tensor products, and the Hermite expansion

Our first concern is with intervals of the form A^p, where $A \in \mathcal{D}_1$. These intervals are more than just *hypercubes*, i.e. intervals $\times_{i=1}^p P^i$ having edges P^i of the same length: the edges themselves are equal. It is a triviality, cf. 1.40(b), that

$$P \in \mathcal{P}_p^{\mathrm{sym}} \iff \exists A \in \mathcal{D}_1 \ni P = A^p. \tag{16.1}$$

The nexus between the η_p and ξ_p measures of symmetric intervals and the *Hermite polynomials* in the Katutani format:[12]

$$\left\{ \begin{array}{l} \forall p \in \mathbb{N}_+, \quad \forall \sigma \in \mathbb{R}_+ \quad \& \quad \forall u \in \mathbb{R}, \\ \qquad H_p(u,\sigma) := \displaystyle\sum_{k=0}^{[p/2]} (-1)^k \binom{p}{2k} \alpha_{2k} \sigma^k u^{p-2k}, \end{array} \right. \tag{16.2}$$

is clear from the next lemma, parts (b), (c):

16.3. Lemma. *Let* $p \in \mathbb{N}_+$ *&* $k \in [1, [p/2]]$. *Then*
(a) $\forall A \in D_1$ & $\forall h \in \mathbb{R}^{p-2k}$,

$$\gamma_k^p(A^p, h) = \binom{p}{2k} \alpha_{2k} \{\ell_1(A)\}^k \chi_{A^{p-2k}}(h);$$

(b) $\forall A \in \mathcal{D}_1$, $\eta_p(A^p) = H_p\{\xi_1(A), \ell_1(A)\}$;
(c) $\forall A \in \mathcal{D}_1$,

$$\xi_p(A^p) = \sum_{k=0}^{[p/2]} \binom{p}{2k} \alpha_{2k} \ell_k(A^k) H_{p-2k}\{\xi_1(A), \ell_1(A)\}.$$

Proof. Let $A \in \mathcal{D}_1$ & $h \in \mathbb{R}^{p-2k}$. Then by 6.17,

$$\gamma_k^p(A^p, h) = \binom{p}{2k} \alpha_{2k} \lambda_{\pi_k}^p(A^p, h), \tag{1}$$

and by (4.19),

$$\lambda_{\pi_k}^p(A^p, h) = \ell_k \Big\{ \mathop{\times}_{\Delta \in \pi_k} A^p(\Delta) \Big\} \chi_{(A^p)_{[2k+1,p]}}(h).$$

But obviously $A^p(\Delta) = A$ and $(A^p)_{[2k+1,p]} = A^{p-2k}$. Hence $\lambda_{\pi_k}^p(A^p, h) = \{\ell_1(A)\}^k \times \chi_{A^{p-2k}}(h)$, and (1) reduces to the equality in (a).

[12] Actually the $H_p(\mu, \sigma)$ defined in (16.2) is $\sqrt{p}!$ times the polynomial in Katutani's paper, cf. (1950, p. 321, (14)).

(*b*) Let $A \in \mathcal{D}_1$ & $k \in [1, [p/2]]$. Integrating the equality in (*a*) with respect to ξ_{p-2k}, we get

$$\int_{\mathbb{R}^{p-2k}} \gamma_k^p(A^p, h)\xi_{p-2k}(\mathrm{d}h) = \binom{p}{2k}\alpha_{2k}\ell_1(A)^k\{\xi_1(A)\}^{p-2k},$$

whence by theorem 15.7 and (16.2),

$$\eta_p(A^p) = \sum_{k=0}^{[p/2]}(-1)^k\int_{\mathbb{R}^{p-2k}} \gamma_k^p(A^p, h)\xi_{p-2k}(\mathrm{d}h) =: H_p\{\xi_1(A), \ell_1(A)\}.$$

Thus (*b*).

(*c*) We now appeal to corollary 11.10(*a*):

$$\begin{aligned}\xi_p(A^p) &= \sum_{k=0}^{[p/2]}\int_{\mathbb{R}^{p-2k}} \gamma_k^p(A^p, h)\eta_{p-2k}(\mathrm{d}h)\\ &= \sum_{k=0}^{[p/2]}\binom{p}{2k}\alpha_{2k}\{\ell_1(A)\}^k\eta_{p-2k}(A^{p-2k}) \qquad \text{by } (a)\\ &= \sum_{k=0}^{[p/2]}\binom{p}{2k}\alpha_{2k}\ell_k(A^k)H_{p-2k}\{\xi_1(A), \ell_1(A)\} \qquad \text{by } (b).\end{aligned}$$

■

Turning to integration, the result on tensor powers that correspond to 16.3(*b*) on sets, reads as follows:

16.4. Theorem. *Let* $f \in L_2(\mathbb{R})$. *Then*

$$\forall p \in \mathbb{N}_+, \qquad f^{\times p} \in L_2(\mathbb{R}^p) \quad \& \quad \mathbb{E}_{\eta_p}(f^{\times p}) = H_p\{\mathbb{E}_{\xi_1}(f), |f|^2_{2,\ell_1}\}.$$

Proof. That $f^{\times p} \in L_2(\mathbb{R}^p)$ follows easily from Fubini's theorem. To turn to the equality, let $p \in \mathbb{N}_+$ and recall that by 12.21(*c*), $\forall k \in [0, [p/2]$ & $\forall h \in \mathbb{R}^{p-2k}$,

$$(f^{\times p})_k^p(h) = \binom{p}{2k}\alpha_{2k}\cdot|f|^{2k}_{2,\ell_1}\cdot f^{\times(p-2k)}(h).$$

Hence by theorem 15.6(*b*),

$$\begin{aligned}\mathbb{E}_{\eta_p}(f^{\times p}) &= \sum_{k=0}^{[p/2]}(-1)^k\mathbb{E}_{\xi_{p-2k}}[(f^{\times p})_k^p]\\ &= \sum_{k=0}^{[p/2]}(-1)^k\binom{p}{2k}\alpha_{2k}|f|^{2k}_{2,\ell_1}\mathbb{E}_{\xi_{p-2k}}[f^{\times(p-2k)}]\\ &= \sum_{k=0}^{[p/2]}(-1)^k\binom{p}{2k}\alpha_{2k}|f|^{2k}_{2,\ell_1}\{\mathbb{E}_{\xi_1}(f)\}^{p-2k}, \qquad \text{by corollary 14.11}\\ &=: H_p\{\mathbb{E}_{\xi_1}(f), |f|^2_{2,\ell_1}\}.\end{aligned}$$

■

On combining this theorem with Ito's factorization theorem 14.20, we get immediately Ito's second important theorem:

16.5. Corollary. (Ito 1951, theorem 3.1) *Let* $f_1, \ldots, f_n \in L_2(\mathbb{R})$ *be* $\perp$. *Then* $\forall p_1, \ldots, p_n \in \mathbb{N}_+$,

$$\mathbb{E}_{\eta_{p_1+\cdots+p_n}}\left(\bigtimes_{i=1}^{n} f_i^{\times p_i}\right) = \prod_{i=1}^{n} H_{p_i}\{\mathbb{E}_{\eta_1}(f_i), |f_i|^2_{2,\ell_1}\}.$$

Proof. By 14.20 and 16.4, we have

$$\text{LHS} = \prod_{i=1}^{n} \mathbb{E}_{\eta_{p_i}}(f_i^{\times p_i}) = \text{RHS}.$$

■

An obvious special case of this, easily proved by taking $f_i = \chi_{D_i}$ and noting that $\eta_1 = \xi_1$, is the following set theoretic analogue of 16.5:

16.6. Corollary. *Let* $D_1, \ldots, D_n \in D_1$ *be* $\|$. *Then* $\forall p_1, \ldots, p_n \in \mathbb{N}_+$,

$$\eta_{p_1+\cdots+p_n}\left(\bigtimes_{i=1}^{n} D_i^{p_i}\right) = \mathbb{E}_{\eta_{p_1+\cdots+p_n}}\left(\bigtimes_{i=1}^{n} \chi_{D_i}^{\times p_i}\right) = \prod_{i=1}^{n} H_{p_i}\{\xi_1(D_i), \ell_1(D_i)\}.$$

16.7. Corollary. (Kakutani 1950, theorem 1) *For all* $p \in \mathbb{N}_{0+}$, $\exists$ *a subspace* $\mathcal{M}_p \subseteq \mathcal{L}_2^{\xi}$ *such that*

$$\mathcal{L}_2^{\xi} = \sum_{p=0}^{\infty} \mathcal{M}_p, \qquad \mathcal{M}_p \perp \mathcal{M}_q, \qquad p \neq q,$$

and $\exists$ *an isometry* W_p *on* $\mathcal{M}_p$ *onto* $L_2^{\text{sym}}(\mathbb{R}^p)$. *Moreover, cf. (1950, p. 322, (17), (18)),* $\forall n \in \mathbb{N}_+$, $\forall \| D_1, \ldots, D_n \in \mathcal{D}_1$ *&* $\forall p_1, \ldots, p_n \in \mathbb{N}_+$ *such that* $p_1 + \cdots + p_n = p$, *we have*

$$W_p\left\{\prod_{i=1}^{n} H_{p_i}\{\xi_1(D_i), \ell_1(D_i)\}\right\} = \sqrt{p!}\bigtimes_{i=1}^{n} \chi_{D_i}^{\times p_i}.$$

Proof. Define $\mathcal{M}_p := \mathcal{S}_{\eta_p}$ & $W_p := \sqrt{p!}\,\mathbb{E}_{\eta_p}^{-1}$. Then by (9.20) and 9.8(c),

$$\mathcal{L}_2^{\xi} = \sum_{p=0}^{\infty} \mathcal{M}_p \quad \& \quad \mathcal{M}_p \perp \mathcal{M}_q, \qquad p \neq q.$$

Also by (10.7),

$$W_p = ((1/\sqrt{p!})\mathbb{E}_{\eta_p})^{-1} \text{ is an isometry on } \mathcal{S}_{\eta_p}, \text{ i.e. on } \mathcal{M}_p, \text{ onto } L_2^{\text{sym}}(\mathbb{R}^p).$$

Finally by the last corollary 16.6,

$$W_p\left\{\prod_{i=1}^{n} H_{p_i}\{\xi_1(D_i), \ell_1(D_i)\}\right\} = \sqrt{p!}\,\mathbb{E}_{\eta_p}^{-1}\left\{\prod_{i=1}^{n} H_{p_i}\{\xi_1(D_i), \ell_1(D_i)\}\right\} = \sqrt{p!}\bigtimes_{i=1}^{n} \chi_{D_i}^{\times p_i}.$$

■

Note. Kakutani's theorem 1 (1950) also asserts that $U_t(\mathcal{M}_p) = \mathcal{M}_p$, where $(U_t, t \in \mathbb{R})$ is the unitary group acting over $\mathcal{L}_2$, induced by the $\mathbb{P}$-measure preserving flow of Brownian motion over Ω. This is easy to show in the context of this paper, the crux being the equality

$$U_t\{\xi_p(D)\} = \xi_p\{D + (t, \dots, t)\}, \quad D \in \mathcal{D}_p, \quad t \in \mathbb{R} \quad \& \quad (t, \dots, t) \in \mathbb{R}^p,$$

which we can prove. Obviously this equality also holds for η_p, and shows that $U_t(\mathcal{S}_{\eta_p}) = \mathcal{S}_{\eta_p}$.

Introducing the abbreviations:

(16.8) $$\forall p \in \mathbb{N}_{0+} \quad \& \quad \forall f \in L_2(\mathbb{R}), \qquad h_p(f) := H_p\{\mathbb{E}_{\xi_1}(f), |f|_{2,\ell_1}\},$$

the equality in 16.4 can be paraphrased as:

(16.9) $$\forall f \in L_2(\mathbb{R}) \quad \& \quad \forall p \in \mathbb{N}_+, \qquad \mathbb{E}_{\eta_p}(f^{\times p}) = h_p(f).$$

Also the equality in 16.5 can be written:

(16.10) $$\begin{cases} \forall \text{ orthogonal sets } \{f_1, \dots, f_p\} \subseteq L_2(\mathbb{R}) \;\&\; \forall p_1, \dots, p_n \in \mathbb{N}_+, \\ \qquad \mathbb{E}_{\eta_{p_1+\cdots+p_n}}\left(\bigtimes_{i=1}^{n} f_i^{\times p_i} \right) = \prod_{i=1}^{n} h_{p_i}(f_i). \end{cases}$$

So far in this paper no orthonormal (o.n.) or other basis has been used, our treatment being coordinate free. We now proceed to show that each o.n. basis in $L_2(\mathbb{R})$ induces, via the Hermite polynomials, an o.n. basis in $\mathcal{S}_{\eta_p}$, for each $p \in \mathbb{N}_+$, and thereby induces an o.n. basis in $\mathcal{L}_2^{\xi}$ itself. This will allow us to deduce from our previously proved results an important theorem first proved by Cameron & Martin (1947, theorem 1).

Since $\forall f_1, \dots, f_p, g_1, \dots, g_p \in L_2(\mathbb{R})$, we obviously have

$$(f_1 \times \cdots \times f_p, g_1 \times \cdots \times g_p)_{L_2(\mathbb{R}^p)} = \prod_{k=1}^{p} (f_k, g_k)_{L_2(\mathbb{R})}.$$

we immediately infer the following:

16.11. Triviality. *Let* $(f_n)_{n=0}^{\infty}$ *be an o.n. basis for* $L_2(\mathbb{R})$. *Then* $\forall p \in \mathbb{N}_+$,

$$((f_{k_1} \times \cdots \times f_{k_p} : (k_1, \dots, k_p) \in \mathbb{N}_{0+}^p) \textit{ is an o.n. basis for } L_2(\mathbb{R}^p)).$$

As to the symmetrization of this basis, it is easily checked that with the f_n as in 16.11, we have

(16.12) $$\begin{cases} \forall (k_1, \dots, k_p) \quad \& \quad (j_1, \dots, j_p) \in \mathbb{N}_{0+}^p, \\ \qquad \text{either } (f_{k_1} \times \cdots \times f_{k_p})\tilde{} = (f_{j_1} \times \cdots \times f_{j_p})\tilde{} \\ \qquad \text{or } (f_{k_1} \times \cdots \times f_{k_p})\tilde{} \perp (f_{j_1} \times \cdots \times f_{j_p})\tilde{}. \end{cases}$$

Also

(16.13) $$\forall (k_1, \dots, k_p) \in \mathbb{N}_{0+}^p, \qquad |(f_{k_1} \times \cdots \times f_{k_p})\tilde{}|_{2,\ell_p}^2 = 1/p!.$$

The upshot of (16.12) and (16.13) and of the fact that for $f = \sum_{j \in J} c_j g_j$, we have $\tilde{f} = \sum_{j \in J} c_j \tilde{g}_j$, is the following lemma:

16.14. Lemma. *Let* $(f_k)_{k=0}^{\infty}$ *be an o.n. basis for* $L_2(\mathbb{R})$ *and* $p \in \mathbb{N}_+$. *Then*

$$(\sqrt{p!}(f_{k_1} \times \cdots \times f_{k_p})\tilde{} : (k_1, \ldots, k_p) \in \mathbb{N}_{0+}^p) \text{ is an o.n. basis for } L_2^{\mathrm{sym}}(\mathbb{R}^p).$$

This basis yields via the isometry $(1/\sqrt{p!})\mathbb{E}_{\eta_p}$ an o.n. basis for $\mathcal{S}_{\eta_k}$. To exhibit this basis conveniently, we shall adopt the following abbreviations:

$$(16.15)\quad \begin{cases} (a) & \forall r \in \mathbb{N}_+, \quad [r] := \{(j_1, \ldots, j_r) : j_1, \ldots, j_r \in \mathbb{N}_{0+} \ \&\ j_1 < \cdots < j_r\}; \\ (b) & \forall r, p \in \mathbb{N}_+, \quad \{r\}_p := \{(p_1, \ldots, p_r) : p_1, \ldots, p_r \in \mathbb{N}_+ \\ & \qquad\qquad\qquad\qquad \&\ p_1 + \cdots + p_r = p\}. \end{cases}$$

The theorem in question then reads:

16.16. Theorem. *Let* $p \in \mathbb{N}_+$ *and* $(f_k)_{k=0}^{\infty}$ *be an o.n. basis for* $L_2(\mathbb{R})$. *Then*

$$\left(\prod_{\alpha=1}^{r} h_{p_\alpha}(f_{j_\alpha}) : r \in [1, p], (j_1, \ldots, j_r) \in [r] \ \&\ (p_1, \ldots, p_r) \in \{r\}_p\right)$$

is an o.n. basis for $\mathcal{S}_{\eta_p}$.

Proof. By (10.7), $(1/\sqrt{p!})\mathbb{E}_{\eta_p}$ is an isometry on $L_2^{\mathrm{sym}}(\mathbb{R}^p)$ onto $\mathcal{S}_{\eta_p}$. Hence from lemma 16.14,

$$(1)\quad \left(\frac{1}{\sqrt{p!}}\mathbb{E}_{\eta_p}[\sqrt{p!}(f_{k_1} \times \cdots \times f_{k_p})\tilde{}] : (k_1, \ldots, k_p) \in \mathbb{N}_{0+}^p\right) \quad \text{is an o.n. basis for } \mathcal{S}_{\eta_p}.$$

Now the $k_1, \ldots, k_p$ in (1) need not be distinct. Let

$$(2)\qquad \mathrm{Range}(k_1, \ldots, k_p) = \{j_1, \ldots, j_r\} \quad \text{where} \quad j_1 < \cdots < j_r,$$

and let $\forall \alpha \in [1, r]$, p_α be the frequency of occurrence of j_α in $(k_1, \ldots, k_p)$, so that

$$(3)\qquad r \in [1, p], \quad p_1, \ldots, p_r \in \mathbb{N}_+ \quad \& \quad p_1 + \cdots + p_r = p.$$

Obviously, $\exists \phi \in \mathrm{Perm}(p) \ni$

$$(k_{\phi(1)}, \ldots, k_{\phi(p)}) = (j_1, \ldots, j_1, j_2, \ldots, j_2, \ldots, j_r, \ldots, j_r),$$

where j_1 is repeated p_1 times, j_2 repeated p_2 times, and so on. It follows that

$$(4)\qquad (f_{k_1} \times \cdots \times f_{k_p})^{\phi^{-1}} = (f_{k_{\phi(1)}} \times \cdots \times f_{k_{\phi(p)}}) = f_{j_1}^{\times p_1} \times \cdots \times f_{j_r}^{\times p_r}.$$

But

$$\begin{aligned} \frac{1}{\sqrt{p!}}\mathbb{E}_{\eta_p}[\sqrt{p!}(f_{k_1} \times \cdots \times f_{k_p})\tilde{}] &= \mathbb{E}_{\eta_p}(f_{k_1} \times \cdots \times f_{k_p}) && \text{by 10.3}(c) \\ &= \mathbb{E}_{\eta_p}\{(f_{k_1} \times \cdots \times f_{k_p})^{\phi^{-1}}\} && \text{by 9.13}(f)\ \&\ \text{A.35}(c) \\ &= \mathbb{E}_{\eta_p}\{f_{j_1}^{\times p_1} \times \cdots \times f_{j_r}^{\times p_r}\} && \text{by (4).} \end{aligned}$$

But by (3), the orthogonality of $f_{j_1}, \ldots, f_{j_p}$, and (16.10), the last integral is

$$\prod_{\alpha=1}^{r} h_{p_\alpha}(f_{j_\alpha}).$$

Thus

$$(5)\qquad \frac{1}{\sqrt{p!}}\mathbb{E}_{\eta_p}[\sqrt{p!}(f_{k_1} \times \cdots \times f_{k_p})\tilde{}] = \prod_{\alpha=1}^{r} h_{p_\alpha}(f_{j_\alpha}).$$

Finally, by (2) and (3),

(6) $$(j_1,\dots,j_r)\in[r] \quad \& \quad (p_1,\dots,p_r)\in\{r\}_p.$$

Substituting from (5) and (6) in (1), we get the desired expression for the basis. ■

16.17. Corollary. *Let* $(f_k)_{k=0}^{\infty}$ *be an o.n. basis for* $L_2(\mathbb{R})$. *Then*

$$\left(\prod_{\alpha=1}^{r} h_{p_\alpha}(f_{j_\alpha}) : r\in\mathbb{N}_+, (j_1,\dots,j_r)\in[r] \;\&\; p_1,\dots,p_r\in\mathbb{N}_+\right)$$

is an o.n. basis for $\mathcal{L}_2^{\xi}$, *cf. definition (8.11).*

Proof. By (9.20) we have the orthogonal decomposition $\mathcal{L}_2^{\xi}=\sum_{p=0}^{\infty}\mathcal{S}_{\eta_p}$. Hence an o.n. basis for $\mathcal{L}_2^{\xi}$ is obtainable by uniting the o.n. bases of all the $\mathcal{S}_{\eta_p}$. Thus from the last theorem we infer that

$$\bigcup_{p=0}^{\infty}\left\{\prod_{\alpha=1}^{r} h_{p_\alpha}(f_{j_\alpha}) : r\in[1,p], (j_1,\dots,j_r)\in[r] \;\&\; (p_1,\dots,p_r)\in\{r\}_p\right\}$$

is an o.n. basis for $\mathcal{L}_2^{\xi}$. But now since p can be any non-negative integer, the conditions $r\in[1,p]$ and $p_1+\dots+p_r=p$ are fulfilled by all $r\in\mathbb{N}_+$. Hence the last basis can be restated as in the enunciation. ■

Expansion in terms of the last o.n. basis yields the following:

16.18. Corollary. (Fourier Hermite series) *Let* (i) $(f_k)_{k=0}^{\infty}$ *be an o.n. basis for* $L_2(\mathbb{R})$, (ii) $\forall x\in\mathcal{L}_2^{\xi}$, $\forall n\in\mathbb{N}_+$, $\forall(j_1,\dots,j_r)\in[r]$ & $\forall p_1,\dots,p_r\in\mathbb{N}_+$,

$$c_{j_1,\dots,j_r}^{p_1,\dots,p_r}(x) := \left(x, \prod_{\alpha=1}^{r} h_{p_\alpha}(f_{j_\alpha})\right)_{\mathcal{L}_2}.$$

Then

$$x=\sum_{r=1}^{\infty}\sum_{(j_1,\dots,j_r)\in[r]}\sum_{p_1=1}^{\infty}\cdots\sum_{p_r=1}^{\infty} c_{j_1,\dots,j_r}^{p_1,\dots,p_r}(x)\cdot\prod_{\alpha=1}^{r} h_{p_\alpha}(f_{j_\alpha}).$$

16.19. *Remarks.* 1. The corollaries 16.17 and 16.18 constitute the theorem 1 of Cameron & Martin (1947). It is deduced from the isometry of $(1/\sqrt{p!})\mathbb{E}_{\eta_p}$, cf. 10.5($c$), and the important equality 16.5 due to Ito (1951, theorem 3.1).

2. In our approach, unlike Ito's (who uses the Cameron–Martin theorem 1 to prove the so-called Wiener–Ito expansion (Ito 1951, theorem 4.2), i.e. prove our corollary 11.2), the o.n. bases and the Cameron–Martin results come at the end. In this respect our approach is close to Wiener's, who too needed the Cameron–Martin results last, in order to develop the theory of the analysis and synthesis of nonlinear transducers (Wiener 1958, lectures 10, 11). This important application of the Cameron–Martin theorem by Wiener has already been discussed in Masani (1966, pp. 113–118), and need not engage us here.

Appendix A. Integrability and integration with respect to a vector measure ρ

In this appendix we review the theory of integrability and integration with respect to a measure ρ with values in a Hilbert space $\mathcal{H}$. In the special case $\mathcal{H}=\mathbb{R}$ or

$\mathcal{H} = \mathbb{C}$, the definitions and results reduce to those of the classical scalar theory, and for the case $\mathcal{H} = \mathcal{L}_2$, they provide what is needed in this paper. We first recall some fundamental ideas of vector-measure theory that we will need. In the sequel

$$\text{(A.1)} \qquad \begin{cases} \mathcal{H} \text{ is a Hilbert space over } \mathbb{R} \text{ or } \mathbb{C}, \\ \mathcal{D} \text{ is a } \delta\text{-ring over a space } \Lambda,\ \rho \in \mathrm{CA}(\mathcal{D}, \mathcal{H}), \\ \mathcal{D}^{\mathrm{loc}} := \{A : A \subseteq \Lambda \ \&\ \forall D \in \mathcal{D}, A \cap D \in \mathcal{D}\}. \end{cases}$$

A.2. *Definition.* A set A is called *ρ-negligible*, in symbols $A \in \mathcal{N}_\rho$, iff

$$A \in \mathcal{D}^{\mathrm{loc}} \quad \& \quad \forall D \in \mathcal{D}, \quad \rho(D \cap A) = 0.$$

C is called a *carrier* of ρ, iff

$$C \in \mathcal{D}^{\mathrm{loc}} \quad \& \quad \forall D \in \mathcal{D}, \quad \rho(D) = \rho(D \cap C).$$

A.3. *Definition.* $\forall A \in \mathcal{D}^{\mathrm{loc}}$, let

$$\Pi_A := \{\pi : \pi \text{ is a finite class of } \| \text{ sets in } \mathcal{D} \cap 2^A\}.$$

(a) The *quasi-variation* $q_\rho(\cdot)$ of ρ is the function on $\mathcal{D}^{\mathrm{loc}}$ defined by $\forall A \in \mathcal{D}^{\mathrm{loc}}$,

$$q_\rho(A) := \sup\{|\rho(\Delta)|_{\mathcal{H}} : \Delta \in \mathcal{D} \cap 2^A\}.$$

(b) The *semi-variation* $s_\rho(\cdot)$ of ρ is the function on $A \in \mathcal{D}^{\mathrm{loc}}$ defined by: $\forall A \in \mathcal{D}^{\mathrm{loc}}$,

$$s_\rho(A) := \sup\left\{\left|\sum_{\Delta \in \pi} \alpha(\Delta)\rho(\Delta)\right|_{\mathcal{H}} : \pi \in \Pi_A, \alpha \in \mathbb{F}^\pi \ \&\ |\alpha(\cdot)| \leqslant 1\right\}.$$

The properties of $q_\rho(\cdot)$ and $s_\rho(\cdot)$ are given in [MN, II, §§ 3.2–3.4]. An especially useful result is

$$\text{(A.4)} \qquad \forall A \in \mathcal{D}^{\mathrm{loc}}, \quad q_\rho(A) \leqslant s_\rho(A) = \sup_{\substack{x \in \mathcal{H}' \\ |x'| \leqslant 1}} |x' \circ \rho|(A) \leqslant 2q_\rho(A),$$

where $|\sigma|(\cdot)$ is the total variation measure of any vector measure $\sigma(\cdot)$. It is easy to see that for $\mathcal{H} = \mathbb{R}$ or $\mathbb{C}$, $q_\rho = s_\rho = |\rho|$. Another useful relation is that

$$\text{(A.5)} \qquad \mathcal{N}_\rho = \{A : A \in \mathcal{D}^{\mathrm{loc}} \ \&\ s_\rho(A) = 0\}.$$

Very important is the family

$$\text{(A.6)} \qquad \mathcal{D}_\rho := \{A : A \in \mathcal{D}^{\mathrm{loc}} \ \&\ s_\rho(A) < \infty\}.$$

A useful lemma is the following, cf. [MN, II, 5.9 and III, C.20]:

A.7. Lemma. (a) *$\mathcal{D} \subseteq \mathcal{D}_\rho \subseteq \mathcal{D}^{\mathrm{loc}} = \mathcal{D}_\rho^{\mathrm{loc}}$ & $\mathcal{D}_\rho$ is a δ-ring.*

(b) *If $(E_n)_1^\infty$ in $\mathcal{D}_\rho$ is such that $\lim_{n\to\infty} E_n$ exists and equals $E \in \mathcal{D}_\rho$, then*

$$\lim_{n\to\infty} s_\rho(E_n) = s_\rho(E) \in \mathbb{R}_{+0};$$

(c) *the following conditions are equivalent:*

(α) $A \in \mathcal{D}_\rho$,

(β) *$A \in \mathcal{D}^{\mathrm{loc}}$ & $\exists \uparrow$ sequence $(\Delta_k)_{k=1}$ in $\mathcal{D} \cap 2^A$ such that*

$$s_\rho(A \backslash \Delta_k) \to 0, \quad \text{as} \quad k \to \infty.$$

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Finally we need a result from general vector measure theory (cf. Dinculeanu 1953, p. 24, Prop. 6):

A.8. *Identity principle.* Let (i) $\mathcal{P}$ be a pre-ring over Λ and $\mathcal{D} = \delta\text{-ring}(\mathcal{P})$, (ii) $\mathcal{X}$ be a Banach space, (iii) $\xi, \eta \in \mathrm{CA}(\mathcal{D}, \mathcal{X})$. Then

$$\xi = \eta \text{ on } \mathcal{P} \quad \Longrightarrow \quad \xi = \eta \text{ on } \mathcal{D}.$$

To turn to integrability, the measure ρ allows us to associate numbers in $[0, \infty]$ with each real-valued $\mathcal{D}^{\mathrm{loc}}$ measurable f on Λ:

$$\text{(A.9)} \quad \begin{cases} \forall f \in \mathcal{M}(\mathcal{D}^{\mathrm{loc}}, \mathcal{B}_1) \quad \& \quad \forall x \in \mathcal{H}', \quad |f|_{1,x'\circ\rho} := \displaystyle\int_\Lambda |f(\lambda)| \cdot |x' \circ \rho| \, (\mathrm{d}\lambda), \\ \forall f \in \mathcal{M}(\mathcal{D}^{\mathrm{loc}}, \mathcal{B}_1), \quad |f|_{1,\rho} := \displaystyle\sup_{\substack{x \in \mathcal{H}' \\ |x'| \leqslant 1}} |f|_{1,x'\circ\rho}, \end{cases}$$

and to define the class of *Gelfand* ρ*-integrable functions*:

$$\text{(A.10)} \qquad \mathcal{G}_{1,\rho} = \{f : f \in \mathcal{M}(\mathcal{D}^{\mathrm{loc}}, \mathcal{B}_1) \ \& \ \forall x' \in \mathcal{H}', \quad |f|_{1,x'\circ\rho} < \infty\}.$$

It is a fundamental result, cf. [MN, II, 3.13(a)] that:

A.11. Theorem. (a) $\mathcal{G}_{1,\rho} = \{f : f \in \mathcal{M}(\mathcal{D}^{\mathrm{loc}}, \mathcal{B}_1) \ \& \ |f|_{1,\rho} < \infty\}$;
(b) $\mathcal{G}_{1,\rho}$ *is a Banach space under the norm* $|\cdot|_{1,\rho}$ *when functions* f, g *in* $\mathcal{G}_{1,\rho}$ *such that* $\mathrm{supp}(f-g) \in \mathcal{N}_\rho$ *are identified;* $f \in \mathcal{G}_{1,\rho} \Longrightarrow |f(\cdot)| \in \mathcal{G}_{1,\rho}$.

Letting for any non-void family $\mathcal{F}$ of subsets of Λ,

$$\text{(A.12)} \qquad S(\mathcal{F}, \mathbb{R}) := \text{the class of } \mathcal{F}\text{-simple functions on } \Lambda \text{ to } \mathbb{R},$$

it follows trivially that

$$\text{(A.13)} \qquad S(\mathcal{D}, \mathbb{R}) \subseteq \mathcal{G}_{1,\rho}.$$

We define the class of *Lebesgue–Pettis*[13] ρ*-integrable* functions by

$$\text{(A.14)} \qquad \mathcal{P}_{1,\rho} := \mathrm{cls}\, S(\mathcal{D}, \mathbb{R}) \quad \text{in} \quad \mathcal{G}_{1,\rho}.$$

The following two results on convergence in $\mathcal{G}_{1,\rho}$, i.e. under the norm $|\cdot|_{1,\rho}$, are crucial (cf. [MN, III, C.2 and C.18]). The first is on the almost-everywhere convergence of subsequences of mean convergent sequences:

A.15. Corollary. *Let* $(f_n)_{n=1}^\infty$ *and* f *be in* $\mathcal{G}_{1,\rho}$ *and* $|f_n - f|_{1,\rho} \to 0$, *as* $n \to \infty$. *Then there exists a subsequence* $(f_{n_k})_{k=1}^\infty$ & $\exists N \in \mathcal{N}_\rho$ *such that*

$$\forall \lambda \in \Lambda \backslash N, \qquad f_{n_k}(\lambda) \to f(\lambda), \quad \text{as} \quad k \to \infty.$$

The second result prescribes conditions sufficient to ensure mean convergence:

A.16. Theorem. *Let* (i) $(f_n)_1^\infty$ *be a sequence in* $\mathcal{P}_{1,\rho}$, (ii) $|f_n(\cdot)| \leqslant g(\cdot) \in \mathcal{P}_{1,\rho}$, *and* (iii) $f_n(\cdot) \to f(\cdot)$ *on* Λ *as* $n \to \infty$. *Then* $|f_n - f|_{1,\rho} \to 0$ *as* $n \to \infty$, *i.e.* f_n *tends to* f *in the Banach space* $\mathcal{G}_{1,\rho}$; *moreover,* $f \in \mathcal{P}_{1,\rho}$.

The previous results depend only on $\mathcal{H}$ being a Banach space. But since $\mathcal{H}$ is in

[13] On the issue of the appropriateness of the appellations 'Lebesgue' or 'Pettis', see [MN, II, remarks 4.15]. As for the lettering $\mathcal{G}$ and $\mathcal{P}$, see footnote 2.

fact a Hilbert space and therefore weakly Σ-complete, we have (cf. Masani & Niemi 1989):

(A.17) $$\mathcal{P}_{1,\rho} = \mathcal{G}_{1,\rho}.$$

From (A.17) follows easily the *Domination Principle*:

(A.18) $$f \in \mathcal{M}(\mathcal{D}^{\text{loc}}, \mathcal{B}_1) \quad \& \quad |f(\cdot)| \leqslant \phi \in \mathcal{P}_{1,\rho} \Longrightarrow f \in \mathcal{P}_{1,\rho}.$$

In particular,

(A.19) $$f \in \mathcal{M}(\mathcal{D}^{\text{loc}}, \mathcal{B}_1) \text{ is bounded } \& \text{ supp } f \subseteq D \in \mathcal{D} \Longrightarrow f \in \mathcal{P}_{1,\xi}.$$

From (A.17) it is also easily seen that

(A.20) $$\mathcal{D}_\rho = \{A : A \in \mathcal{D}^{\text{loc}} \ \& \ \chi_A \in \mathcal{P}_{1,\rho}\}, \qquad \text{cf. (A.6).}$$

This in turn reveals the nature of $\mathcal{D}_\rho$-simple integrable functions:

(A.21) $$\mathcal{S}(\mathcal{D}^{\text{loc}}, \mathbb{R}) \cap \mathcal{P}_{1,\rho} = \mathcal{S}(\mathcal{D}_\rho, \mathbb{R}).$$

We can now attend to the approximation of integrable functions by $\mathcal{D}_\rho$ and $\mathcal{D}$ simple ones. From (A.21) and the $\mathcal{D}^{\text{loc}}$-simple approximatability of $\mathcal{D}^{\text{loc}}$-measurable functions, we easily get:

A.22. Triviality. **($\mathcal{D}_\rho$-simple approximation)** *Let* $f \in \mathcal{P}_{1,\rho}$. *Then* $\exists$ *a sequence* $(s_n)_1^\infty$ *in* $\mathcal{S}(\mathcal{D}_\rho, \mathbb{R})$ *such that*

$$s_n(\cdot) \to f(\cdot) \quad \& \quad |s_n(\cdot)| \uparrow |f(\cdot)| \quad \text{on} \quad \Lambda \quad \& \quad \lim_{n\to\infty} |s_n - f|_{1,\rho} = 0.$$

To turn to approximation by $\mathcal{D}$-simple functions, the mean convergence asserted in theorem A.24(a) below is obvious from (A.14). But the simultaneous fulfillment of both statements asserted in A.24 requires careful proof. The following preliminary result is needed.

A.23. Lemma. *Let* $f \in \mathcal{S}(\mathcal{D}_\rho, \mathbb{R})$. *Then* $\exists$ *a sequence* $(g_p)_1^\infty$ *in* $\mathcal{S}(\mathcal{D}, \mathbb{R})$ *such that* $\forall p \in \mathbb{N}_+$, $|g_p(\cdot)| \leqslant |f(\cdot)|$ *on* Λ, *and* $|g_p - f|_{1,\rho} < 1/p$.

Proof. Let $f = \sum_{k=1}^r a_k \chi_{A_k}$, in standard form. Let $M := \sum_{k=1}^r |a_k|$. Then each $A_k \in \mathcal{D}_\rho$. Hence by lemma A.7(c), $\forall p \in \mathbb{N}_+$, $\exists D_k^p \in \mathcal{D} \cap 2^{A_k}$ such that

(1) $$\forall k \in [1, r], \quad s_\rho(A_k \backslash D_k^p) < 1/(Mp).$$

Define

(2) $$\forall p \in \mathbb{N}_+, \quad g_p := \sum_{k=1}^r a_k \chi_{D_k^p}.$$

Then obviously $g_p \in \mathcal{S}(\mathcal{D}, \mathbb{R})$ and $|g_p(\cdot)| \leqslant |f(\cdot)|$ on Λ. Also, since $f - g_p = \sum_{k=1}^r a_k \chi_{A_k \backslash D_k^p}$, therefore by (1),

$$|f - g_p|_{1,\rho} \leqslant \sum_{k=1}^r |a_k| \cdot |\chi_{A_k \backslash D_k^p}|_{1,\rho} = \sum_{k=1}^r |a_k| s_\rho(A_k \backslash D_k^p) \leqslant \frac{1}{p}.$$

■

A.24. Theorem. **($\mathcal{D}$-simple function approximation)** *Let* $f \in \mathcal{P}_{1,\rho}$. *Then* $\exists$ *a sequence* $(s_n)_{n=1}^\infty$ *in* $\mathcal{S}(\mathcal{D}, \mathbb{R})$ *such that*

(a) $\forall n \geqslant 1$, $|s_n(\cdot)| \leqslant |f(\cdot)|$ *on* Λ $\quad \& \quad$ $\lim_{n\to\infty} |s_n - f|_{1,\rho} = 0$;

(*b*) $\exists N \in \mathcal{N}_\rho$ *such that* $\forall \lambda \in \Lambda \backslash N$, $\lim_{n\to\infty} s_n(\lambda) = f(\lambda)$.

Proof. Since $f \in \mathcal{M}(\mathcal{D}^{\text{loc}}, \mathcal{B}_1)$, therefore $\exists (f_n)_1^\infty$ in $S(\mathcal{D}^{\text{loc}}, B_1)$ such that

(1) $$f_n(\cdot) \to f(\cdot) \quad \& \quad |f_n(\cdot)| \uparrow |f(\cdot)| \quad \text{on} \quad \Lambda.$$

It follows from (A.18) that each $f_n \in \mathcal{P}_{1,\rho}$ and from theorem A.16 that

(2) $$|f_n - f|_{1,\rho} \to 0, \quad \text{as} \quad n \to \infty.$$

The equality (A.21) now tells us that

(3) $$\text{each } f_n \in S(\mathcal{D}_\rho, \mathbb{R}).$$

It follows from lemma A.23 that

(4) $$\begin{cases} \forall p \in \mathbb{N}_+, \quad \exists (g_n^p)_{n=1}^\infty \quad \text{in} \quad S(\mathcal{D}, \mathbb{R}) \ni \forall n \in \mathbb{N}_+, \\ \& \qquad\qquad |g_n^p(\cdot)| \leqslant |f_n(\cdot)|, \quad |g_n^p - f_n|_{1,\rho} < 1/2p. \end{cases}$$

Now let $p \in \mathbb{N}_+$. Then by (2), $\exists n_p \in \mathbb{N}_+$ such that

$$|f_{n_p} - f|_{1,\rho} < 1/2p.$$

Hence by (4),

$$|g_{n_p}^p - f|_{1,\rho} \leqslant |g_{n_p}^p - f_{n_p}|_{1,\rho} + |f_{n_p} - f|_{1,\rho} \leqslant 1/p.$$

Also by (4), $g_{n_p}^p \in S(\mathcal{D}, \mathbb{R})$, and by (4) and (1),

$$|g_{n_p}^p(\cdot)| \leqslant |f_{n_p}(\cdot)| \leqslant |f(\cdot)|.$$

Letting $g_p := g_{n_p}^p$, we have thus shown that

(5) $$\exists (g_p)_{p=1}^\infty \quad \text{in} \quad S(\mathcal{D}, \mathbb{R}) \ni \forall p \geqslant 1, \quad |g_p(\cdot)| \leqslant |f(\cdot)| \quad \& \quad |g_p - f|_{1,\rho} < 1/p.$$

It follows from corollary A.15 that $\exists N \in \mathcal{N}_\rho$ and $\exists$ a subsequence $(g_{p_n})_{n=1}^\infty$ such that

(6) $$\forall \lambda \in \Lambda \backslash N, \qquad g_{p_n}(\lambda) \to f(\lambda) \quad \text{as} \quad n \to \infty.$$

Defining $\forall n \in \mathbb{N}_+$, $s_n(\cdot) = g_{p_n}(\cdot)$, it is clear from (5) and (6) that the sequence $(s_n)_{n=1}^\infty$ has both the properties (*a*), (*b*). ■

We turn next to integration with respect to ρ. This is to be understood as an operator $\mathbb{E}_\rho$ on the Banach space $\mathcal{P}_{1,\rho}$ to the Hilbert space $\mathcal{H}$. The definition of $\mathbb{E}_\rho$ is in two steps. First:

(A.25) $$\forall s := \sum_{k=1}^r a_k \chi_{D_k} \in S(\mathcal{D}, \mathbb{R}), \qquad \mathbb{E}_\rho(s) := \sum_{k=1}^r a_k \rho(D_k) \in \mathcal{H}.$$

It follows easily that $\mathbb{E}_\rho$ is a linear operator on the linear manifold $S(\mathcal{D}_p, \mathbb{R})$ to $\mathcal{H}$, which is a contraction, i.e.

$$\forall s \in S(\mathcal{D}, \mathbb{R}), \qquad |\mathbb{E}_\rho(s)|_{\mathcal{H}} \leqslant |s|_{1,\rho}.$$

Hence $\mathbb{E}_\rho$ extends to a linear contraction on the cls $S(\mathcal{D}_p, \mathbb{R})$, i.e. on $\mathcal{P}_{1,\rho}$. We denote this extension by the same symbol; thus

A.26. *Definition.* $\forall f \in \mathcal{P}_{1,\rho}$, $\mathbb{E}_\rho(f) := \lim_{n\to\infty} \mathbb{E}_\rho(s_n)$, where $(s_n)_{n=1}^\infty$ is any sequence in $S(\mathcal{D}, \mathbb{F})$ such that $|s_n - f|_{1,\rho} \to 0$, as $n \to \infty$. We also define the integral by

$$\int_{\mathbb{R}^p} f(t)\rho(\mathrm{d}t) := \mathbb{E}_\rho(f) \in \mathcal{H}.$$

We of course have

(A.27) $$\forall f \in \mathcal{P}_{1,\rho}, \qquad |\mathbb{E}_\rho(f)|_{\mathcal{H}} \leqslant |f|_{1,\rho}.$$

The most important result on integration is the following (cf. [MN, II, 4.5(c), III, C.18]):

A.28. Theorem. (On dominated convergence) *Let*

(i) $(f_n)_{n=1}^{\infty}$ *be a sequence in* $\mathcal{P}_{1,\rho}$,

(ii) $\forall n \in \mathbb{N}_+$, $|f_n(\cdot)| \leqslant g(\cdot) \in \mathcal{P}_{1,\rho}$,

(iii) $\exists N \in \mathcal{N}_\rho \ni \lim_{n\to\infty} f_n(\cdot) = f(\cdot)$ *on* $\Lambda \backslash N$.

Then

$$f \in \mathcal{P}_{1,\rho}, \quad \lim_{n\to\infty} |f_n - f|_{1,\rho} = 0 \quad \& \quad \mathbb{E}_\rho(f) = \lim_{n\to\infty} \mathbb{E}_\rho(f_n).$$

Another which we shall need relates to the linear manifold in $\mathcal{H}$ spanned by the range of the measure ρ and the range of the linear operator $\mathbb{E}_\rho$. They have the same closure (cf. [MN, Part II, 4.6]):

(A.29) $$\operatorname{cls} \operatorname{Range} \mathbb{E}_\rho = \mathcal{S}_\rho := \operatorname{cls}\langle \operatorname{Range} \rho \rangle.$$

For two vector measures we have the following simple result:

A.30. Triviality. *Let* $\rho, \sigma \in \mathrm{CA}(\mathcal{D}, \mathcal{H})$. *Then*

(a) $\forall f \in \mathcal{M}(\mathcal{D}^{\mathrm{loc}}, \mathcal{B}_1), \quad |f|_{1,\rho+\sigma} \leqslant |f|_{1,\rho} + |f|_{1,\sigma}$;

(b) $\mathcal{P}_{1,\rho} \cap \mathcal{P}_{1,\sigma} \subseteq \mathcal{P}_{1,\rho+\sigma}$;

(c) $\forall f \in \mathcal{P}_{1,\rho} \cap \mathcal{P}_{1,\sigma}, \quad \mathbb{E}_{p+\sigma}(f) = \mathbb{E}_\rho(f) + \mathbb{E}_\sigma(f)$.

Proof. (a) and (b) follow easily on applying the inequality for the total variation, to wit $|\mu+\nu|(\cdot) \leqslant |\mu|(\cdot) + |\nu|(\cdot)$, taking $\mu := x' \circ \rho$, $\nu := x' \circ \sigma$, $x' \in \mathcal{H}'$. The equality in (c) obviously prevails for $\mathcal{D}$-simple f, and by a limiting argument can be shown to hold for all $f \in \mathcal{P}_{1,\rho} \cap \mathcal{P}_{1,\sigma}$. ■

The inclusion in A.30(b) sharpens to an equality when the ranges of the measures ρ and σ are orthogonal.

A.31. Proposition. *Let* $\rho, \sigma \in \mathrm{CA}(\mathcal{D}, \mathcal{H})$ & $\mathcal{S}_\rho \perp \mathcal{S}_\sigma$. *Then*

(a) $\forall f \in \mathcal{M}(\mathcal{D}^{\mathrm{loc}}, \mathcal{B}_1), \ \max\{|f|_{1,\rho}, |f|_{1,\sigma}\} \leqslant |f|_{1,\rho+\sigma}$;

(b) $\mathcal{P}_{1,\rho+\sigma} = \mathcal{P}_{1,\rho} \cap \mathcal{P}_{1,\sigma}$.

Proof. (a) Let $f \in \mathcal{M}(\mathcal{D}^{\mathrm{loc}}, \mathcal{B}_1)$, $x' \in \mathcal{H}'$ & $|x'| \leqslant 1$. Then $\exists_1 x \in \mathcal{H}$ such that

(1) $$|x| \leqslant 1 \quad \& \quad \forall y \in \mathcal{H}, \quad x'(y) = (y, x)_{\mathcal{H}}.$$

Since $\mathcal{S}_\rho \perp \mathcal{S}_\sigma$, we see that

(2) $$x = x_0 + x_1 + x_2 \quad \text{where} \quad x_0 \perp \mathcal{S}_\rho + \mathcal{S}_\sigma, \quad x_1 \in \mathcal{S}_\rho \quad \& \quad x_2 \in \mathcal{S}_\sigma.$$

Let $x_i' \in \mathcal{H}'$ correspond to x_i, $i = 0, 1, 2$, and grant momentarily that

(I) $$|x' \circ \rho|(\cdot) = |x_1' \circ \rho|(\cdot) \quad \text{on} \quad \mathcal{D}^{\mathrm{loc}}.$$

(II) $$\forall x \in \mathcal{S}_\rho, \quad |x' \circ \rho|(\cdot) = |x' \circ (\rho + \sigma)|(\cdot) \quad \text{on} \quad \mathcal{D}^{\mathrm{loc}}.$$

Then

(3) $$|f|_{1,x'\circ\rho} = |f|_{1,x_1'\circ\rho} \quad \& \quad \forall x \in \mathcal{S}_\rho, \qquad |f|_{1,x'\circ\rho} = |f|_{1,x'\circ(\rho+\sigma)}.$$

Since by (2), $x_1 \in \mathcal{S}_\rho$, it follows from (3) that

$$|f|_{1,x'\circ\rho} = |f|_{1,x'_1\circ\rho} = |f|_{1,x'_1\circ(\rho+\sigma)}$$
$$\leqslant |x'_1| \cdot |f|_{1,\rho+\sigma} = |x_1| \cdot |f|_{1,\rho+\sigma} \leqslant |x| \cdot |f|_{1,\rho+\sigma} \leqslant |f|_{1,\rho+\sigma}.$$

This holds for any $x \in \mathcal{H}'$ such that $|x'| \leqslant 1$. Hence taking the supremum for $|x'| \leqslant 1$ on the LHS, we get

$$|f|_{1,\rho} \leqslant |f|_{1,\rho+\sigma}.$$

Similarly, $|f|_{1,\sigma} \leqslant |f|_{1,\rho+\sigma}$. This establishes (a) except for the justification of (I) and (II), which we shall leave to the reader.

(b) This follows readily on combining the inequalities in (a) and in A.30(a). ■

The integration $\mathbb{E}_\rho$ has of course the following 'Pettis property' (as can be easily shown). Let $\mathcal{K}$ be a Hilbert space over $\mathbb{F}$ and $T \in \mathrm{CL}(\mathcal{H}, \mathcal{K})$. Then

$$\text{(A.32)} \quad \begin{cases} T \circ \rho \in \mathrm{CA}(\mathcal{D}, \mathcal{K}) \ \&\ \forall f \in \mathcal{P}_{1,\rho}, \ \ f \in \mathcal{P}_{1,T\circ\rho} \ \ \&\ \ T\{\mathbb{E}_\rho(f)\} = \mathbb{E}_{T\circ\rho}(f), \\ \qquad \forall f \in \mathcal{P}_{1,\rho} \ \ \&\ \ \forall x' \in \mathcal{H}', \ \ x'\{\mathbb{E}_\rho(f)\} = \mathbb{E}_{x'\circ\rho}(f). \end{cases}$$

Now in the special case of interest, $\mathcal{H} = \mathcal{L}_2$, it is a fundamental fact stemming from the simple inequality

$$\forall x \in \mathcal{L}_2, \quad |\mathbb{E}_{\mathbb{P}}(x)| \leqslant |x|_{\mathcal{L}_1} \leqslant |x|_{\mathcal{L}_2},$$

that $\mathbb{E}_{\mathbb{P}} \in (\mathcal{L}_2)'$. Hence from (A.32) we conclude that

$$\text{(A.33)} \qquad \text{when } \mathcal{H} = \mathcal{L}_2, \qquad \forall f \in \mathcal{P}_{1,\rho}, \qquad \mathbb{E}_{\mathbb{P}}\{\mathbb{E}_\rho(f)\} = \mathbb{E}_{\mathbb{E}_{\mathbb{P}}\circ\rho}(f).$$

We turn next to the invariance of Lebesgue Pettis ρ-integrability and ρ-integration under ρ-measure-preserving transformations of Λ into Λ.

A.34. Triviality. *Let* (i) Λ, $\mathcal{D}$ *be as in (A.1),* (ii) ϕ *be one–one on* Λ *onto* Λ *and such that*

$$\forall D \in \mathcal{D}, \quad \phi^{-1}(D) \ \&\ \phi(D) \in \mathcal{D}.$$

Then

$$\phi \ \&\ \phi^{-1} \in \mathcal{M}(\mathcal{D}^{\mathrm{loc}}, \mathcal{D}^{\mathrm{loc}}).$$

We leave the simple proof to the reader.

A.35. Lemma. (Measure-preserving transformation) *Let*

(i) Λ, $\mathcal{D}$ & ρ *be as in (A.1),*

(ii) ϕ *be one–one on* Λ *onto* Λ *such that* $\forall D \in \mathcal{D}$, $\phi^{-1}(D)$ & $\phi(D) \in \mathcal{D}$,

(iii) ϕ & ϕ^{-1} *be* ρ*-measure-preserving, i.e.*

$$\forall D \in \mathcal{D}, \qquad \rho\{\phi^{-1}(D)\} = \rho(D) = \rho\{\phi(D)\}.$$

Then

(a) $\forall x' \in \mathcal{H}'$, ϕ *is* $|x' \circ \rho|$ *measure-preserving, i.e.*

$$\forall A \in \mathcal{D}^{\mathrm{loc}}, \qquad |x' \circ \rho|\{\phi^{-1}(A)\} = |x' \circ \rho|(A);$$

(b) $\forall f \in \mathcal{M}(\mathcal{D}^{\mathrm{loc}}, \mathcal{B}_1)$, $\qquad f \in \mathcal{P}_{1,\rho}$ *iff* $f \circ \phi \in \mathcal{P}_{1,\rho}$;

(c) $\forall f \in \mathcal{P}_{1,\rho}$, $\mathbb{E}_\rho(f \circ \phi) = \mathbb{E}_\rho(F)$.

Proof. (a) Let $x' \in \mathcal{H}'$ and $\mu = x' \circ \rho \in \mathrm{CA}(\mathcal{D}, \mathbb{R})$. Then by (iii)

$$\text{(1)} \qquad \forall A \in \mathcal{D}^{\mathrm{loc}}, \qquad \mu\{\phi^{-1}(D)\} = \mu(D) = \mu\{\phi(D)\}.$$

Now let $A \in \mathcal{D}^{\mathrm{loc}}$. Then by A.34, $\phi^{-1}(A) \in \mathcal{D}^{\mathrm{loc}}$. Now clearly $\pi = \{\Delta_1, \dots, \Delta_r\}$ is a $\mathcal{D}$-partition of A if and only if $\pi' = \{\phi^{-1}(\Delta_1), \dots, \phi^{-1}(\Delta_r)\}$ is a $\mathcal{D}$-partition of $\phi^{-1}(A)$. Hence by (1)

$$\sum_{\Delta' \in \pi'} \mu(\Delta') = \sum_{\Delta \in \pi} \mu(\Delta).$$

Taking the suprema, we get $|\mu|\{\phi^{-1}(A)\} = |\mu|(A)$, i.e. we have (a).

(b) It follows from (a) classically that $\forall f \in \mathcal{M}(\mathcal{D}^{\mathrm{loc}}, \mathcal{B}_1)$,

$$\int_\Lambda |(f \circ \phi)(t)| \cdot |x' \circ \rho| \,(\mathrm{d}t) = \int_\Lambda |f(t)| \cdot |x' \circ \rho| \,(\mathrm{d}t),$$

whence, cf. (A.9), $|f \circ \phi|_{1,\rho} = |f|_{1,\rho}$. This entails by (A.17) and (A.10) the equivalence (b).

(c) Let $f \in \mathcal{P}_{1,\rho}$, $x' \in \mathcal{H}'$ and $\mu := x' \circ \rho$. Then $f \in L_{1,\mu}$. It therefore follows classically from (1) that

$$\int_\Lambda f\{\phi(\lambda)\}(x' \circ \rho) \,(\mathrm{d}\lambda) = \int_\Lambda f(\lambda)(x' \circ \rho) \,(\mathrm{d}\lambda).$$

As this holds $\forall x' \in \mathcal{H}'$, we have (c). ■

The concepts of ρ-integrability and ρ-integration allow us to define for each $\mathcal{D}^{\mathrm{loc}}$ measurable $\mathbb{R}$-valued function f on Λ, an indefinite integral $\int_A f(t)\rho\,(\mathrm{d}t)$ for suitable sets A. The formal definitions are as follows:

A.36. *Definition.* Let $f \in \mathcal{M}(\mathcal{D}^{\mathrm{loc}}, \mathcal{B}_1)$. Then
(a) $\mathcal{D}_\rho(f) := \{A : A \in \mathcal{D}^{\mathrm{loc}} \;\&\; f\chi_A \in \mathcal{P}_{1,\rho}\}$;
(b) $\forall A \in \mathcal{D}_\rho(f)$, $\nu_{\rho,f}(A) := \mathbb{E}_\rho(f\chi_A)$.
The measure $\nu_{\rho,f}$ is called *the indefinite integral of f with respect to ρ.*

As basic results, easily proved, we have

$$\text{(A.37)} \qquad \begin{cases} \mathcal{D}_\rho(f) \text{ is a } \delta\text{-ring} \subseteq \mathcal{D}^{\mathrm{loc}} \quad \& \quad [\mathcal{D}_\rho(f)]^{\mathrm{loc}} = \mathcal{D}^{\mathrm{loc}}, \\ \forall f \in \mathcal{P}_{1,\rho}, \quad \mathcal{D}_\rho(f) = \mathcal{D}^{\mathrm{loc}}, \\ \nu_{\rho,f}(A) \in \mathrm{CA}(\mathcal{D}_\rho(f), \mathcal{H}). \end{cases}$$

A result of considerable importance is the following on the semi-variation of the indefinite integral:

A.38. Theorem. $\forall f \in \mathcal{M}(\mathcal{D}^{\mathrm{loc}}, \mathcal{B}_1)$, $s_{\nu_{\rho,f}}(\Lambda) = |f|_{1,\rho} \in [0, \infty)$.

Proof. Let $f \in \mathcal{M}(\mathcal{D}^{\mathrm{loc}}, \mathcal{B}_1)$. Then by the equality in (A.4),

$$\text{(1)} \qquad s_{\nu_{\rho,f}}(\Lambda) = \sup_{\substack{x \in \mathcal{H}' \\ |x'| \leqslant 1}} |x' \circ \nu_{\rho,f}|(\Lambda).$$

Now let $x' \in \mathcal{H}'$. Then by definitions A.36(b) and (A.32),

$$\text{(2)} \qquad \forall D \in \mathcal{D}_\rho(f), \qquad (x' \circ \nu_{\rho,f})(D) = \int_\Lambda \chi_D(\lambda) \cdot f(\lambda) \cdot (x' \circ \rho) \,(\mathrm{d}\lambda).$$

It follows from (2) and a basic result on scalar measures (cf. [MN, I, 2.32(a)]), that

$$|x' \circ \nu_{\rho,f}|(\Lambda) = \int_\Lambda |f(\lambda)| \cdot |x' \circ \rho| \,(\mathrm{d}\lambda) \in [0, \infty].$$

Substituting in (1), we get, cf. (A.9)

$$s_{\nu_{\rho,f}}(\Lambda) = \sup_{\substack{x\in\mathcal{H}' \\ |x'|\leqslant 1}} \int_\Lambda |f(\lambda)| \cdot |x' \circ \rho| (\mathrm{d}\lambda) =: |f|_{1,\rho}.$$

■

This theorem yields the following corollary on the quasi- and semi-variations, which is very useful for our purposes in § 10:

A.39. Corollary. *Let* $f \in \mathcal{M}(\mathcal{D}^{\text{loc}}, \mathcal{B}_1)$. *Then*

$$\sup_{C\in\mathcal{D}_\rho(f)} |\mathbb{E}_\rho(f\chi_C)| \leqslant |f|_{1,\rho} \leqslant 2 \sup_{C\in\mathcal{D}_\rho(f)} |\mathbb{E}_\rho(f\chi_C)|.$$

Proof. Equation (A.4) tells us that

$$(1) \qquad q_{\nu_{\rho,f}}(\Lambda) \leqslant s_{\nu_{\rho,f}}(\Lambda) \leqslant 2q_{\nu_{\rho,f}}(\Lambda).$$

But by A.38, the middle term is $|f|_{1,\rho}$. Also, since $\nu_{\rho,f} \in \mathrm{CA}(\mathcal{D}_\rho(f), \mathbb{R}_{0+})$, therefore by A.3($a$),

$$q_{\nu_{\rho,f}}(\Lambda) = \sup_{C\in\mathcal{D}_\rho(f)} |\nu_{\rho,f}(C)| := \sup_{C\in\mathcal{D}_\rho(f)} |\mathbb{E}_\rho(f\chi_C)|.$$

Thus (1) reduces to the desired inequalities. ■

The indefinite integral $\nu_{\rho,f}$ in the special case where f is constantly 1 on Λ is of measure-theoretic importance. From definition A.36 and (A.20) we see at once that

$$(\text{A.40}) \qquad \begin{cases} \text{for } f = 1 \text{ on } \Lambda, & \mathcal{D}_\rho(f) := \{A : A \in \mathcal{D}^{\text{loc}} \ \&\ \chi_A \in \mathcal{P}_{1,\rho}\} = \mathcal{D}_\rho \\ \& \quad \forall A \in \mathcal{D}_\rho(f), & \nu_{\rho,f}(A) = \mathbb{E}_\rho(\chi_A). \end{cases}$$

For brevity we write $\bar{\rho}(A)$ instead of $\nu_{\rho,1}(A)$. Obviously, cf. (A.37),

$$(\text{A.41}) \qquad \rho \subseteq \bar{\rho} \in \mathrm{CA}(\bar{\mathcal{D}}_\rho, \mathcal{H}).$$

whence

$$(\text{A.42}) \qquad \forall B \in \mathcal{D}_\rho \cap \sigma\text{-ring}(\mathcal{D}), \qquad D_n \in \mathcal{D} \ \&\ D_n \uparrow B \Longrightarrow \bar{\rho}(B) = \lim_{n\to\infty} \rho(D_n).$$

It can be shown that $\bar{\rho}$ is the maximal CA extension that ρ admits. In the very special case that $\rho = \mu \in \mathrm{CA}(\mathcal{D}, \mathbb{R}_{0+})$ it easily follows that

$$(\text{A.43}) \qquad \mu \subseteq \bar{\mu} = \mathrm{Rstr.}_{\mathcal{D}_\mu}|\mu| \subseteq |\mu|.$$

The next and final proposition is needed in the study of the Fubini theorem. Let $\mathcal{C}$, $\mathcal{D}$ be δ-rings over the sets S and T, $\rho \in \mathrm{CA}(\mathcal{C}, \mathcal{L}_2)$ and $\sigma \in \mathrm{CA}(\mathcal{D}, \mathcal{L}_2)$. Let

$$\forall C \times D \in \mathcal{C} \times \mathcal{D}, \qquad (\rho \times \sigma)(C \times D) := \rho(C) \cdot \sigma(D).$$

Then in general $(\rho \times \sigma)(C \times D) \notin \mathcal{L}_2$ & $\rho \times \sigma \notin \mathrm{CA}(\mathcal{C} \times \mathcal{D}, \mathcal{L}_2)$. For the ρ and σ encountered in this paper, it turns out that $\rho \times \sigma$ is CA and extends to the δ-ring$(\mathcal{C} \times \mathcal{D})$. For such ρ, σ, the following result on the connection between the negligibility classes $\mathcal{N}_\rho$, $\mathcal{N}_\sigma$, $\mathcal{N}_{\rho\times\sigma}$ is useful.

A.44. Proposition. *Let* (i) $\mathcal{C}$, $\mathcal{D}$ *be* δ*-rings over* S *and* T,
(ii) $\rho \in \mathrm{CA}(\mathcal{C}, \mathcal{L}_2)$, $\sigma \in \mathrm{CA}(\mathcal{D}, \mathcal{L}_2)$,
(iii) $\rho \times \sigma \in \mathrm{CA}\{\delta\text{-}ring(\mathcal{C} \times \mathcal{D}), \mathcal{L}_2\}$.

Then $(\mathcal{N}_\rho \times \mathcal{D}^{\mathrm{loc}}) \cup (\mathcal{C}^{\mathrm{loc}} \times \mathcal{N}_\sigma) \subseteq \mathcal{N}_{\rho\times\sigma}$.

Proof. It will suffice to show that $\mathcal{N}_\rho \times \mathcal{D}^{\mathrm{loc}} \subseteq \mathcal{N}_{\rho\times\sigma}$, i.e. that

$$\text{(I)} \qquad \forall N \in \mathcal{N}_\rho \quad \& \quad \forall B \in \mathcal{D}^{\mathrm{loc}}, \qquad N \times B \in \mathcal{N}_{\rho\times\sigma}.$$

Proof of (I). Let $\hat{\mathcal{R}} = \mathrm{ring}(\mathcal{C} \times \mathcal{D})$ and $\hat{\mathcal{D}} = \delta\text{-ring}(\mathcal{C} \times \mathcal{D})$ and let

$$\text{(1)} \qquad N \in \mathcal{N}_\rho \quad \& \quad B \in \mathcal{D}^{\mathrm{loc}}.$$

$$\text{(2)} \qquad \mathcal{F} := \{F : F \in \hat{\mathcal{D}} \ \& \ (\rho \times \sigma)[F \cap (N \times B)] = 0\}.$$

We leave it to the reader to check that

$$\text{(A)} \qquad \hat{\mathcal{R}} \subseteq \mathcal{F} = \text{a } \delta\text{-monotone class}.$$

Then as usual, $\mathcal{F} = \hat{\mathcal{D}}$, i.e. by (2),

$$\forall E \in \hat{\mathcal{D}}, \qquad (\rho \times \sigma)[E \cap (N \times B)] = 0.$$

Thus, cf. definition A.2, $N \times B \in \mathcal{N}_{\rho\times\sigma}$. This establishes (I) and shows that $\mathcal{N}_\rho \times \mathcal{D}^{\mathrm{loc}} \subseteq \mathcal{N}_{\rho\times\sigma}$. ■

Appendix B. Integration with respect to measures given by Markovian kernels

Let $\mathcal{X}$ be a Banach space over $\mathbb{R}$, $p, q \in \mathbb{N}_+$, $\sigma \in \mathrm{CA}(\mathcal{D}_q, \mathcal{X})$, and let the kernel $K(\cdot,\cdot)$ on $\mathcal{D}_p \times \mathbb{R}^q$ to $\mathbb{R}_{0+}$ be 'Markovian' in the wide sense that (i) for each $h \in \mathbb{R}^q$, $K(\cdot, h) \in \mathrm{CA}(\mathcal{D}_p, \mathbb{R}_{0+})$,[14] and (ii) for each $D \in \mathcal{D}_p$, $K(D, \cdot) \in \mathcal{P}_{1,\sigma}$. It then easily follows that

$$\text{(1)} \qquad \rho(\cdot) := \int_{\mathbb{R}^q} K(\cdot, h)\sigma\,(\mathrm{d}h) \in \mathrm{CA}(\mathcal{D}_p, \mathcal{X}).$$

The question arises as to what precisely is the $\mathcal{P}_{1,\rho}$ class and whether the integration $\mathbb{E}_\rho$ obeys the equation:

$$\text{(2)} \qquad \forall f \in \mathcal{P}_{1,\rho}, \qquad \mathbb{E}_\rho(f) = \int_{\mathbb{R}^q} \left\{ \int_{\mathbb{R}^p} f(t) K\,(\mathrm{d}t, h) \right\} \sigma\,(\mathrm{d}h).$$

The implication on (1) $\Longrightarrow$ (2) does prevail when the kernel $K(\cdot,\cdot)$ is an indefinite integral, i.e. when

$$\text{(3)} \qquad \forall h \in \mathbb{R}^q, \qquad K(D, h) = \int_D k(t, h)\mu\,(\mathrm{d}t),$$

where $\mu \in \mathrm{CA}(\mathcal{D}_p, \mathbb{R}_{0+})$ and $k(\cdot,\cdot)$ is on $\mathbb{R}^p \times \mathbb{R}^q$ to $\mathbb{R}_{0+}$. For then, from (1), we get

$$\begin{aligned} \rho(D) &= \int_{\mathbb{R}^q} \left\{ \int_D k(t, h)\mu\,(\mathrm{d}t) \right\} \sigma\,(\mathrm{d}h) \\ \text{(4)} \qquad &= \int_D \left\{ \int_{\mathbb{R}^q} k(t, h)\sigma\,(\mathrm{d}h) \right\} \mu\,(\mathrm{d}t), \end{aligned}$$

by the vectorial Fubini theorem for the product measure $\sigma \times \mu$ in [MN, III, 9.8(d)],

[14] In ordinary Markoff theory, $K(\cdot, h)$ is required to be a probability measure (cf. Doob 1953, pp. 255, 613).

the RHS(4) being the Pettis integral of an $\mathcal{X}$-valued function. From (4) it moreover follows by a substitution principle proved in [MN, IV (unpublished)] that

$$\begin{aligned}\mathbb{E}_\rho(f) &= \int_{\mathbb{R}^p} f(t)\left\{\int_{\mathbb{R}^q} k(t,h)\sigma(\mathrm{d}h)\right\}\mu(\mathrm{d}t)\\ &= \int_{\mathbb{R}^q}\left\{\int_{\mathbb{R}^p} f(t)k(t,h)\mu(\mathrm{d}t)\right\}\sigma(\mathrm{d}h),\end{aligned}$$

where the last equality again hails from the same Fubini theorem. By virtue of (3) this can of course be written in the form (2). Thus with the stipulation (3) on $K(\cdot,\cdot)$, the implication (1) $\Longrightarrow$ (2) emerges as a corollary of the vectorial Fubini and substitution theorems.

The objective of this appendix is (i) to show that the implication (1) $\Longrightarrow$ (2) is valid without any imposition on $K(\cdot,\cdot)$ beyond its wide sense Markovianness, provided that the vector measure σ is subject to a rather stringent restraint. A related objective is (ii) to identify the class $\mathcal{P}_{1,\rho}$ when σ is so restrained. The restraint in question is so stringent that most scalar and vector measures violate it. But, as we shall show at the end of this appendix, the important measure η_q appearing in this paper satisfies it, and this appendix is indispensable for the integration theory of Wiener's measure ξ_p.

Before we state the restraint on σ, it is convenient to dispose of the following preliminary lemma, which validates the implication (1) $\Longrightarrow$ (2) for a non-negative measure σ on $\mathcal{B}_q$, and for integrals which can take the value ∞.

B.1. Preliminary lemma. *Let* (i) $p,q\in\mathbb{N}_+$, (ii) $\Lambda(\cdot,\cdot)$ *be a 'Markovian' kernel on* $\mathcal{B}_p\times\mathbb{R}^q$, *i.e.*

$$\begin{aligned}\forall h\in\mathbb{R}^q,&\qquad \Lambda(\cdot,h)\in\mathrm{CA}(\mathcal{B}_p,[0,\infty]),\\ \forall A\in\mathcal{B}_p,&\qquad \Lambda(A,\cdot)\in\mathcal{M}(\mathcal{B}_q,\mathcal{B}_{[0,\infty]}),\end{aligned}$$

(iii) $\mu\in\mathrm{CA}(\mathcal{B}_q,[0,\infty])$, *and* (iv)

$$A\in\mathcal{B}_p,\qquad \nu(A):=\int_{\mathbb{R}^q}\Lambda(A,h)\mu(\mathrm{d}h).$$

Then (*a*) $\nu\in\mathrm{CA}(\mathcal{B}_p,[0,\infty])$*;*

(*b*) $\forall f\in\mathcal{M}(\mathcal{B}_p,\mathcal{B}_{[0,\infty]})$ *(cf. p. 1185 for* $\mathcal{B}_{[0,\infty]}$*)*

$$\int_{\mathbb{R}^p} f(t)\nu(\mathrm{d}t)=\int_{\mathbb{R}^q}\left\{\int_{\mathbb{R}^p} f(t)\Lambda(\mathrm{d}t,h)\right\}\mu(\mathrm{d}h)\in[0,\infty].$$

Proof. (*a*) Let $\forall k\in\mathbb{N}_+$, $A_k\in\mathcal{B}_p$ be $\|$ and $A=\bigcup_{k=1}^\infty A_k$. Then by (ii), $\forall h\in\mathbb{R}^q$, $\Lambda(A,h)=\sum_{k=1}^\infty\Lambda(A_k,h)$, whence by (iv),

$$\nu(A)=\int_{\mathbb{R}^q}\left\{\sum_{k=1}^\infty\Lambda(A_k,h)\right\}\mu(\mathrm{d}h)=\sum_{k=1}^\infty\nu(A_k).$$

Thus (*a*).

(*b*) First let $f=\sum_{i=1}^r a_i\chi_{A_i}\in S(\mathcal{B}_p,\mathbb{R}_{0+})$. Then by (iv),

$$\mathbb{E}_\nu(f)=\sum_{i=1}^r a_i\nu(A_i)=\sum_{i=1}^r a_i\int_{\mathbb{R}^q}\Lambda(A_i,h)\mu(\mathrm{d}h)=\int_{\mathbb{R}^q}\left\{\sum_{i=1}^r a_i\Lambda(A_i,h)\right\}\mu(\mathrm{d}h).$$

Since the integrand is $\int_{\mathbb{R}^p} f(t)\Lambda(\mathrm{d}t, h)$, we have the desired equality for simple f.

Next, let $f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}(\mathbb{R}_{0+}))$. Then there exists a sequence $(s_n)_{n=1}^{\infty}$ in $\mathcal{S}(\mathcal{B}_p, \mathcal{B}(\mathbb{R}_{0+}))$ such that $s_n \uparrow f$ on $\mathbb{R}^p$. Then a straightforward argument involving three applications of the monotone convergence theorem yields (b). ■

We now state the restraint on σ. Guided by the uses we have in mind, we shall take $\mathcal{X}$, the range of σ, to be a Hilbert space. Accordingly, we shall formulate the restraint as follows:

$$\text{(B.2)} \qquad \begin{cases} \mathcal{H} \text{ is a Hilbert space over } \mathbb{R} \\ \sigma \in \mathrm{CA}(\mathcal{D}_q, \mathcal{H}) \text{ is such that } \exists c \in \mathbb{R}_+ \ni \\ \qquad\qquad \forall g \geqslant 0 \text{ in } \mathcal{P}_{1,\sigma}, \quad |g|_{1,\sigma} \leqslant c|\mathbb{E}_\sigma(g)|_{\mathcal{H}}. \end{cases}$$

The properties we assign to the Markovian kernels $K(\cdot,\cdot)$ are the following, where p, $q \in \mathbb{N}_+$:

$$\text{(B.3)} \qquad \begin{cases} \text{(i) } K(\cdot,\cdot) \text{ is a function on } \mathcal{D}_p \times \mathbb{R}^q \text{ to } \mathbb{R}_{0+}, \\ \text{(ii) } \forall h \in \mathbb{R}^q,\ K(\cdot, h) \in \mathrm{CA}(\mathcal{D}_p, \mathbb{R}_{0+}), \\ \text{(iii) } \forall D \in \mathcal{D}_p,\ K(D,\cdot) \in \mathcal{P}_{1,\sigma}, \text{ where } \sigma \text{ is as in (B.2)}, \\ \text{(iv) } \forall D \in \mathcal{D}_p^{\mathrm{sym}},\ K(D,\cdot) \text{ is symmetric on } \mathbb{R}^q \text{ to } \mathbb{R}_{0+}. \end{cases}$$

The symbol $|K|(\cdot, h)$ will denote the total variation measure of the measure $K(\cdot, h)$. The analogues of the properties (B.2), (ii)–(iv), for $|K|(\cdot,\cdot)$ are given in the next result, the simple proof of which we leave to the reader:

B.4. Triviality. *Let $K(\cdot,\cdot)$ be as in (B.3), and $\forall h \in \mathbb{R}^q$, $|K|(\cdot, h)$ be the total variation measure of $K(\cdot, h)$. Then*

(*a*) $|K|(\cdot,\cdot)$ *is a function on $\mathcal{B}_p \times \mathbb{R}^q$ to $[0,\infty]$;*
(*b*) $\forall h \in \mathbb{R}^q$, $|K|(\cdot, h) \in \mathrm{CA}(\mathcal{B}_p, [0,\infty])$;
(*c*) $\forall B \in \mathcal{B}_p$, $|K|(B,\cdot) \in \mathcal{M}(\mathcal{B}_q, \mathcal{B}_{[0,\infty]})$;
(*d*) $\forall B \in \mathcal{B}_p^{\mathrm{sym}}$, $|K|(B,\cdot)$ *is symmetric on $\mathbb{R}^q$ to $[0,\infty]$.*

Our derivation of (2) from (1) is heavily dependent on the extension $\bar{\rho}$ of ρ on the augmented δ-ring $(\mathcal{D}_p)_\rho$, cf. A.6. These two entities must therefore first engage our attention. For them we have the:

B.5. Main lemma. *Let*

(i) $\mathcal{H}$, σ *be as in (B.2) and $K(\cdot,\cdot)$ be as in (B.3),*
(ii) $\forall D \in \mathcal{D}_p$, $\rho(D) := \int_{\mathbb{R}^q} K(D, h)\sigma(\mathrm{d}h)$.

Then

(*a*) $\rho \in \mathrm{CA}(\mathcal{D}_p, \mathcal{H})$;
(*b*) $\forall x' \in \mathcal{H}'$ & $\forall B \in \mathcal{B}_p$,

$$|x' \circ \rho|(B) \leqslant \int_{\mathbb{R}^q} |K|(B,h) \cdot |x' \cdot \sigma|\,(\mathrm{d}t) \in [0,\infty];$$

(*c*) $(\mathcal{D}_p)_\rho = \{B : B \in \mathcal{B}_p \ \&\ |K|(B,\cdot) \in \mathcal{P}_{1,\sigma}\}$;
(*d*) $\forall B \in (\mathcal{D}_p)_\rho$, $\bar{\rho}(B) = \int_{\mathbb{R}^q} |K|(B,h)\sigma(\mathrm{d}h)$.

Proof. (*a*) From (B.3)(ii), and (ii), it is clear that

$$(1) \qquad \rho \in \mathrm{FA}(\mathcal{D}_p, \mathcal{H}).$$

Next let $\forall n \in \mathbb{N}_+$, $D_n \in \mathcal{D}_p$ & $D_n \downarrow \emptyset$. Then by (B.3)(ii), (iii), $\forall h \in \mathbb{R}^q$,

$$(2) \qquad K(D_n, h) \downarrow 0 \quad \& \quad \forall n \in \mathbb{N}_+, \qquad K(D_n, \cdot) \leqslant K(D_1, \cdot) \in \mathcal{P}_{1,\sigma}.$$

By (2) and the dominated convergence theorem A.28,

$$(3) \qquad \rho(D_n) := \int_{\mathbb{R}^q} K(D_n, h)\sigma\,(\mathrm{d}h) \to \int_{\mathbb{R}^q} 0\sigma\,(\mathrm{d}h) = 0, \quad \text{as} \quad n \to \infty.$$

By (1), (3) and the Kolmogorov condition, we have (*a*).

(*b*) Let $x' \in \mathcal{H}'$ and first let $D \in \mathcal{D}_p$. Then

$$|(x' \circ \rho)(D)| = \left| \int_{\mathbb{R}^q} K(D, h)(x' \circ \sigma)\,(\mathrm{d}h) \right|$$

whence it follows easily that

$$|x' \circ \rho|(D) \leqslant \int_{\mathbb{R}^q} K(D, h)|x' \circ \sigma|\,(\mathrm{d}h).$$

Next let $B \in \mathcal{B}_p$ and $D_n \in \mathcal{D}_p$ & $D_n \uparrow B$. Then, using the monotone convergence theorem:

$$|x' \circ \rho|(B) = \lim_{n\to\infty} |x' \circ \rho|(D_n) \leqslant \int_{\mathbb{R}^q} |K|(B, h) \cdot |x' \circ \sigma|\,(\mathrm{d}h) \leqslant \infty.$$

Thus (*b*).

(*c*) Let $B \in \mathcal{B}_p$. Then we have to show that

$$(\mathrm{I}) \qquad B \in (\mathcal{D}_p)_\rho \iff |K|(B, \cdot) \in \mathcal{P}_{1,\sigma}.$$

Proof of (I). Let $B \in (\mathcal{D}_p)_\rho$. Then by (A.6), $s_\rho(B) < \infty$, and by (A.42), for D_n in $\mathcal{D}_p$ such that $D_n \uparrow B$, we have

$$(4) \qquad \bar{\rho}(B) = \lim_{n\to\infty} \rho(D_n) \in \mathcal{H}.$$

Also,

$$(5) \qquad |K|(B, h) = \lim_{n\to\infty} K(D_n, h) \in [0, \infty].$$

Now by (B.3), each $K(D_n, \cdot) \geqslant 0$ and $K(D_n, \cdot) \in \mathcal{P}_{1,\sigma}$. Hence by the crucial (B.2),

$$|K(D_n, \cdot)|_{1,\sigma} \leqslant c \cdot |\mathbb{E}_\sigma\{K(D_n, \cdot\}| =: c|\rho(D_n)|.$$

Letting $n \to \infty$ in this, it follows from (5) and (4) that $||K|(B, \cdot)|_{1,\sigma} \leqslant c|\bar{\rho}(B)| < \infty$, since $B \in (\mathcal{D}_p)_\rho$. Thus by A.11(*a*) and (A.17),

$$|K|(B, \cdot) \in \mathcal{P}_{1,\sigma}.$$

Next, let $\mathcal{B} \in \mathcal{B}_p$ and $|K|(B, \cdot) \in \mathcal{P}_{1,\sigma}$. Then taking the $\sup_{|x'|\leqslant 1}$ in the two terms in (*b*), we get

$$s_\rho(B) \leqslant ||K|(B, \cdot)|_{1,\sigma} < \infty, \quad \text{since} \quad |K|(B, \cdot) \in \mathcal{P}_{1,\sigma}.$$

Hence $B \in (\mathcal{D}_p)_\rho$. This establishes (I) and proves (*c*).

(*d*) Let $B \in (\mathcal{D}_p)_\rho$ and let, as in (4), $D_n \uparrow B$. Then, cf. A.42.

$$(6) \qquad \bar{\rho}(B) = \lim_{n\to\infty} \rho(D_n) := \lim_{n\to\infty} \int_{\mathbb{R}^q} K(D_n, h)\sigma\,(\mathrm{d}h).$$

By (5), $K(D_n, \cdot) \to |K|(B, \cdot)$ & $\forall n \in \mathbb{N}_+$, $|K(D_n, \cdot)| \leqslant |K|(D_n, \cdot) \leqslant |K|(B, \cdot)$. But by

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(c), $|K|(B,\cdot) \in \mathcal{P}_\sigma$. Hence by the dominated convergence theorem A.28, (6) reduces to

$$\bar{\rho}(B) = \int_{\mathbb{R}^p} |K|(B,h)\sigma(\mathrm{d}h).$$

Thus (d). ■

We turn next to integration with respect to the measure ρ under consideration. Paraphrased in terms of integration, the results B.5(c), (d) read:

$$\forall B \in \mathcal{B}_p, \qquad \chi_B \in \mathcal{P}_{1,\rho} \iff \int_{\mathbb{R}^p} \chi_B(t)K(\mathrm{d}t,\cdot) \in \mathcal{P}_{1,\sigma},$$

and when $\chi_B \in \mathcal{P}_{1,\sigma}$,

$$\int_{\mathbb{R}^p} \chi_B(t)\rho(\mathrm{d}t) = \int_{\mathbb{R}^q} |K|(B,h)\sigma(\mathrm{d}h) = \int_{\mathbb{R}^q} \left\{ \int_{\mathbb{R}^p} \chi_B(t)K(\mathrm{d}t,h) \right\}\sigma(\mathrm{d}h).$$

Our goal is to show, cf. B.8, that this result generalizes to arbitrary measurable f. It is convenient to first dispose of the case of $(\mathcal{D}_p)_\rho$ simple functions f:

B.6. Lemma. **($(\mathcal{D}_p)_\rho$ simple f)** *Let* (i) K *and* ρ *be as in* (B.3) *and* (B.5)(ii), *and* (ii) $f \in \mathsf{S}\{(\mathcal{D}_p)_\rho, \mathbb{R}\}$. *Then*

(a) $\hat{f}(\cdot) := \int_{\mathbb{R}^p} f(t)K(\mathrm{d}t,\cdot) \in \mathcal{P}_{1,\sigma}$;

(b) $\int_{\mathbb{R}^p} f(t)\rho(\mathrm{d}t) = \int_{\mathbb{R}^q} \hat{f}(h)\sigma(\mathrm{d}h)$;

(c) *when* f *is symmetric on* $\mathbb{R}^p$, $\hat{f}(\cdot)$ *is symmetric on* $\mathbb{R}^q$.

Proof. (a) We first assert that

$$\text{(I)} \qquad \forall B \in (\mathcal{D}_p)_\rho \quad \& \quad \forall h \in \mathbb{R}^q, \qquad \int_{\mathbb{R}^p} \chi_B(t)K(\mathrm{d}t,h) = |K|(B,h) \in [0,\infty].$$

Proof of (I). It is a fundamental fact that for any non-negative CA measure ν on a δ-ring $\mathcal{D}$, $\nu \subseteq |\nu| \in \mathrm{CA}(\mathcal{D}^{\mathrm{loc}},[0,\infty])$, and that $\mathbb{E}_\nu = \mathbb{E}_{|\nu|}$. Applying this to $\nu = K(\cdot,h)$ for $h \in \mathbb{R}^q$, we have $\forall h \in \mathbb{R}^q$,

$$\int_{\mathbb{R}^p} \chi_B(t)K(\mathrm{d}t,h) = \int_{\mathbb{R}^p} \chi_B(t)|K|(\mathrm{d}t,h) = |K|(B,h).$$

Thus (I).

Now let $f = \sum_{i=1}^r a_i\chi_{A_i} \in \mathsf{S}((\mathcal{D}_p)_\rho, \mathbb{R})$. Then by (I),

$$\text{(1)} \qquad \hat{f}(\cdot) := \int_{\mathbb{R}^p} f(t)K(\mathrm{d}t,\cdot) = \sum_{i=1}^r a_i|K|(A_i,\cdot).$$

Since $A_i \in (\mathcal{D}_p)_\rho$, therefore by the B.5($c$), each $|K|(A_i,\cdot)$ is in $\mathcal{P}_{1,\sigma}$. Hence so is the sum $\hat{f}(\cdot)$. Thus (a).

(b) We have

$$\begin{aligned}
\int_{\mathbb{R}^p} f(t)\rho(\mathrm{d}t) &= \sum_{i=1}^r a_i \int_{\mathbb{R}^p} \chi_{A_i}(t)\rho(\mathrm{d}t) = \sum_{i=1}^r a_i\bar{\rho}(A_i) \\
&= \sum_{i=1}^r a_i \int_{\mathbb{R}^q} |K|(A_i,h)\sigma(\mathrm{d}h), \qquad \text{by B.5}(d) \\
&= \int_{\mathbb{R}^q} \left\{ \sum_{i=1}^r a_i|K|(A_i,h) \right\}\sigma(\mathrm{d}h) = \int_{\mathbb{R}^q} \hat{f}(h)\sigma(\mathrm{d}h), \qquad \text{by (1).}
\end{aligned}$$

Thus (b).

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(*c*) When f is symmetric, then by (1.45), we can take each $A_i \in \mathcal{B}_{\hat{p}}^{\mathrm{sym}}$. Hence by B.4(*d*), each $|K|(A_i, \cdot)$ is symmetric on $\mathbb{R}^q$, and therefore by (1), so is $\hat{f}(\cdot)$. Thus *c*. ■

It is also convenient to note the equivalence of the following conditions valid for any measurable f:

B.7. Triviality. *Let* K *and* ρ *be as in* (B.3) *and* B.5(ii). *Then the following conditions are equivalent:*

(α) $f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$ &

$$\sup_{\substack{x' \in \mathcal{H}' \\ |x'| \leqslant 1}} \int_{\mathbb{R}^q} \left\{ \int_{\mathbb{R}^p} |f(t)| \cdot K\,(\mathrm{d}t, h) \right\} |x' \circ \sigma| < \infty,$$

(β) $f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1)$ & $\exists N \in \mathcal{N}_\sigma \ni \forall h \in \mathbb{R}^q \backslash N$,

$$f \in L_{1, K\,(\cdot, h)} \quad \& \quad \int_{\mathbb{R}^p} |f(t)| K\,(\mathrm{d}t, \cdot) \in \mathcal{P}_{1,\sigma}.$$

Proof. Let (α) hold, and write

$$\forall h \in \mathbb{R}^q, \qquad \Phi(h) := \int_{\mathbb{R}^p} |f(t)| K\,(\mathrm{d}t, h) \in [0, \infty]$$

and

$$N := \{h : h \in \mathbb{R}^q \ \& \ \Phi(h) = \infty\}.$$

Then (α) asserts that

$$\sup_{\substack{x' \in \mathcal{H}' \\ |x'| \leqslant 1}} \int_{\mathbb{R}^q} \Phi(h) |x' \circ \sigma|\,(\mathrm{d}h) < \infty,$$

and from this it easily follows that $s_\sigma(N) = 0$, i.e. $N \in \mathcal{N}_\sigma$. Thus $\exists N \in \mathcal{N}_\sigma$ such that $\forall h \in \mathbb{R}^q \backslash N$,

(1) $$\int_{\mathbb{R}^p} |f(t)| K\,(\mathrm{d}t, h) = \Phi(h) < \infty, \quad \text{i.e.} \quad f \in L_{1, K\,(\cdot, h)}.$$

Moreover, (α) tells us that $|\Phi|_{1,\sigma} < \infty$, i.e.

(2) $$\int_{\mathbb{R}^p} |f(t)| K\,(\mathrm{d}t, \cdot) = \Phi(\cdot) \in \mathcal{P}_{1,\sigma}, \qquad \text{by A.11}(a) \text{ and A.17}.$$

By (1) and (2), we have (β).

Next let (β) hold. Then (2) holds. Hence by A.17 and A.11(a),

$$\infty > \left| \int_{\mathbb{R}^p} |f(t)| K\,(\mathrm{d}t, \cdot) \right|_{1,\sigma} := \sup_{\substack{x' \in \mathcal{H}' \\ |x'| \leqslant 1}} \int_{\mathbb{R}^q} \left| \int_{\mathbb{R}^p} |f(t)| K\,(\mathrm{d}t, h) \right| |x' \circ \sigma|\,(\mathrm{d}h),$$

i.e. we have (α). ■

B.8. Main theorem. *Let* (i) $p, q \in \mathbb{N}_+$, (ii) *the measure* σ *be as in* (B.2) *and the kernel* $K\,(\cdot, \cdot)$ *as in* (B.3), (iii)

$$\forall D \in \mathcal{D}_p, \qquad \rho\,(D) := \int_{\mathbb{R}^q} K(D, h) \sigma\,(\mathrm{d}h).$$

Then

(a)

$$\mathcal{P}_{1,\rho} = \Big\{ f : f \in \mathcal{M}(\mathcal{B}_p, \mathcal{B}_1) \ \&\ \exists N \in \mathcal{N}_\sigma \ni \forall h \in \mathbb{R}^q \backslash N,$$
$$f \in L_{1,K(\cdot,h)} \ \&\ \int_{\mathbb{R}^p} |f(t)| K(\mathrm{d}t, \cdot) \in \mathcal{P}_{1,\sigma} \Big\};$$

(b) *for* $f \in \mathcal{P}_{1,\rho}$ & N *as in* (a), *letting* $\forall h \in \mathbb{R}^q \backslash N$, $\hat{f}(h) := \int_{\mathbb{R}^p} f(t) K(\mathrm{d}t, h)$, *we have*

$$\hat{f} \in \mathcal{P}_{1,\sigma} \quad \& \quad \mathbb{E}_\sigma(\hat{f}) = \mathbb{E}_\rho(f);$$

(c) *when* σ *is permutation-invariant on* $\mathcal{D}_q$,

$$f \text{ is symmetric on } \mathbb{R}^p \Longrightarrow \exists \bar{N} \in \mathcal{N}_\sigma^{\mathrm{sym}} \ni \hat{f}(\cdot) \text{ is symmetric on } \mathbb{R}^q \backslash \bar{N}.$$

Proof. (a) $\subseteq$. Notice that the RHS(a) $= \{f : f$ satisfies B.7(β)$\}$.

Case 1. Let $f \in \mathcal{P}_{1,\rho}$ & $f(\cdot) \geqslant 0$. Then by A.21, there exist $(s_n)_{n=1}^\infty$ in $\mathcal{S}\{(\mathcal{D}_p)_\rho, \mathbb{R}_{0+}\}$ such that

(1) $$0 \leqslant s_n(\cdot) \uparrow f(\cdot) \quad \text{on} \quad \mathbb{R}^p.$$

Hence by the dominated convergence theorem A.28, and B.6(b),

$$\mathbb{E}_\rho(f) = \lim_{n\to\infty} \mathbb{E}_\rho(s_n) = \lim_{n\to\infty} \mathbb{E}_\sigma(\hat{s}_n).$$

Now since $s_n > 0$ and $K(\cdot,\cdot) \geqslant 0$, therefore

$$\hat{s}_n(\cdot) := \int_{\mathbb{R}^p} s_n(t) K(\mathrm{d}t, \cdot) \geqslant 0.$$

Also by B.6(a), $\hat{s}_n \in \mathcal{P}_{1,\sigma}$. Hence by the crucial (B.2), $|\hat{s}_n|_{1,\sigma} \leqslant c|\mathbb{E}_\sigma(\hat{s}_n)|$. Thus

(2) $$\overline{\lim_{n\to\infty}} |\hat{s}_n|_{1,\sigma} \leqslant c \overline{\lim_{n\to\infty}} |\mathbb{E}_\rho(\hat{s}_n)| = c|\mathbb{E}_\sigma(f)|.$$

To evaluate the LHS(2), note that by (1) and the monotone convergence theorem, as $n \to \infty$,

$$\hat{s}_n(h) := \int_{\mathbb{R}^p} s_n(t) K(\mathrm{d}t, h) \uparrow \int_{\mathbb{R}^p} |f|(t) K(\mathrm{d}t, h), \qquad h \in \mathbb{R}^q.$$

Hence for a fixed x' with $|x'| \leqslant 1$, again by the monotone convergence theorem,

$$\begin{aligned} &\int_{\mathbb{R}^q} \Big\{ \int_{\mathbb{R}^p} f(t) K(\mathrm{d}t, h) \Big\} |x' \circ \sigma| (\mathrm{d}h) \\ &\qquad = \lim_{n\to\infty} \int_{\mathbb{R}^q} \hat{s}_n(h) |x' \circ \sigma| (\mathrm{d}h) \\ &\qquad \leqslant \overline{\lim_{n\to\infty}} \sup_{\substack{x' \in \mathcal{H}' \\ |x'| \leqslant 1}} \int_{\mathbb{R}^q} |\hat{s}_n(h)| \cdot |x' \circ \sigma| (\mathrm{d}h) = \overline{\lim_{n\to\infty}} |\hat{s}_n|_{1,\sigma} \\ &\qquad \leqslant c|\mathbb{E}_\sigma(f)|, \qquad \text{by (2)}. \end{aligned}$$

Taking the supremium for $|x'| \leqslant 1$ on the LHS, we get

$$\sup_{\substack{x' \in \mathcal{H}' \\ |x'| \leqslant 1}} \int_{\mathbb{R}^q} \Big\{ \int_{\mathbb{R}^p} f(t) K(\mathrm{d}t, h) \Big\} |x' \circ \sigma| (\mathrm{d}h) \leqslant c|\mathbb{E}_\sigma(f)|.$$

But since $f \in \mathcal{P}_{1,\rho}$, the RHS $< \infty$. Thus f satisfies B.7(α), and therefore the equivalent B.7(β). Thus $f \in \mathrm{RHS}(a)$.

Case 2. Let $f \in \mathcal{P}_{1,\rho}$. Then by (A.15), $|f(\cdot)| \in \mathcal{P}_{1,\rho}$ and $|f(\cdot)| \geqslant 0$. Hence by Case 1,

$$\sup_{\substack{x' \in \mathcal{H}' \\ |x'| \leqslant 1}} \int_{\mathbb{R}^q} \left\{ \int_{\mathbb{R}^p} |f(t)| K(\mathrm{d}t, h) \right\} |x' \circ \sigma| (\mathrm{d}h) \leqslant c\infty,$$

i.e. B.7(α) holds, and as before, $f \in \mathrm{RHS}(a)$. This completes the proof of $\subseteq$.

Proof. (a) $\supseteq$. Let $f \in \mathrm{RHS}(a)$. Then f satisfies B.7(β). Thus there exists $N \in \mathcal{N}_\sigma$ such that $\forall h \in \mathbb{R}^q \backslash N$,

$$f \in L_{1,K(\cdot,h)} \quad \& \quad \int_{\mathbb{R}^p} |f(t)| K(\mathrm{d}t, \cdot) \in \mathcal{P}_{1,\sigma}.$$

It follows from (A.17) and A.11(a) that

$$\beta = \sup_{\substack{x' \in \mathcal{H}' \\ |x'| \leqslant 1}} \int_{\mathbb{R}^q} \left\{ \int_{\mathbb{R}^p} |f(t)| K(\mathrm{d}t, h) \right\} |x' \circ \sigma| (\mathrm{d}h) < \infty. \tag{3}$$

To show that $f \in \mathcal{P}_{1,\rho}$, it will by (A.17) and A.10 suffice to show that

$$\forall x' \in \mathcal{H}', \qquad \int_{\mathbb{R}^p} |f(t)| \cdot |x' \circ \rho| (\mathrm{d}t) < \infty. \tag{I}$$

Proof of (I). Let $x' \in \mathcal{H}'$. Then by (iii),

$$\forall D \in \mathcal{D}_p, \qquad (x' \circ \rho)(D) = \int_{\mathbb{R}^q} K(D, h)(x' \circ \sigma)(\mathrm{d}h),$$

whence by a routine argument,

$$\forall B \in \mathcal{B}_p, \qquad |x' \circ \rho|(B) \leqslant \int_{\mathbb{R}^q} |K|(B,h)|(x' \circ \sigma)| (\mathrm{d}h) =: \mu_{x'}(B), \quad \text{say}. \tag{4}$$

It follows from (4) that

$$\int_{\mathbb{R}^p} |f(t)| \cdot |x' \circ \rho| (\mathrm{d}t) \leqslant \int_{\mathbb{R}^p} |f(t)| \mu_{x'} (\mathrm{d}t) < \infty. \tag{5}$$

But by the definition of $\mu_{x'}$ in (4) and lemma B.1(b),

$$\mathrm{RHS}(5) = \int_{\mathbb{R}^q} \left\{ \int_{\mathbb{R}^p} |f(t)| \cdot |K|(\mathrm{d}t, h) \right\} |x' \circ \sigma| (\mathrm{d}h) \leqslant \beta < \infty, \qquad \text{by (3)}.$$

Thus (5) reduces to (I). This finishes the proof of (a).

(b) With $f \in \mathcal{P}_{1,\rho}$ and N as in (a), let $\forall h \in \mathbb{R}^q \backslash N$,

$$\hat{f}(h) := \int_{\mathbb{R}^p} f(t) K(\mathrm{d}t, h). \tag{5$'$}$$

First note that by B.6(a), (b),

$$\forall s \in S((\mathcal{D}_p)_\rho, \mathbb{R}), \qquad \hat{s} \in \mathcal{P}_{1,\sigma} \quad \& \quad \mathbb{E}_\rho(s) = \mathbb{E}_\sigma(\hat{s}). \tag{6}$$

Now since $f \in \mathcal{P}_{1,\rho}$, therefore by theorem A.22, there exist a sequence $(s_n)_{n=1}^\infty$ in $S((\mathcal{D}_p)_\rho, \mathbb{R})$ such that

$$s_n(\cdot) \to f(\cdot) \quad \& \quad |s_n(\cdot)| \uparrow |f(\cdot)| \quad \text{on} \quad \mathbb{R}^p. \tag{7}$$

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By (7), the dominated convergence theorem A.28 and (6),

$$\mathbb{E}_\rho(f) = \lim_{n\to\infty} \mathbb{E}_\rho(s_n) = \lim_{n\to\infty} \mathbb{E}_\sigma(\hat{s}_n). \tag{8}$$

Now let $h \in \mathbb{R}^q \backslash N$, where N is as in (a). Then since by (a), $f \in L_{1,K(\cdot,h)}$, therefore by (7) and Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty} \hat{s}_n(h) := \lim_{n\to\infty} \int_{\mathbb{R}^p} s_n(t) K\,(\mathrm{d}t, h) = \int_{\mathbb{R}^q} f(t) K\,(\mathrm{d}t, h) = \hat{f}(h). \tag{9}$$

Moreover, by (7) and (a),

$$|\hat{s}_n(\cdot)| := \left| \int_{\mathbb{R}^p} s_n(t) K\,(\mathrm{d}t, \cdot) \right| \leqslant \int_{\mathbb{R}^p} |s_n(t)| K\,(\mathrm{d}t, \cdot)$$

$$\leqslant \int_{\mathbb{R}^p} |f(t)| K\,(\mathrm{d}t, \cdot) \in \mathcal{P}_{1,\sigma}. \tag{10}$$

By (9), (10) and the dominated convergence theorem A.28,

$$\mathbb{E}_\sigma(\hat{f}) = \lim_{n\to\infty} \mathbb{E}_\sigma(\hat{s}_n) = \mathbb{E}_\rho(f), \qquad \text{by (8)}.$$

Thus (b).

(c) Let f be symmetric on $\mathbb{R}^p$. Then by 1.44, we may assume that the s_n in (7) are symmetric. A repetition of (9), which recall holds for $h \in \mathbb{R}^q \backslash N$, yields

$$\exists N \in \mathcal{N}_\sigma \ni \forall h \in \mathbb{R}^q \backslash N, \qquad \hat{f}(h) = \lim_{n\to\infty} \hat{s}_n(h), \qquad \text{by } (5'). \tag{11}$$

Since σ is permutation invariant, $\forall \psi \in \mathrm{Perm}(q)$, $N^\psi \in \mathcal{N}_\sigma$, and so

$$N \subseteq \tilde{N} := \bigcup_{\psi \in \mathrm{Perm}(q)} N^\psi \in \mathcal{N}_\sigma^{\mathrm{sym}}.$$

Since $\mathbb{R}^q \backslash \tilde{N}$ is symmetric, we conclude that $\forall \psi \in \mathrm{Perm}(q)$,

$$h \in \mathbb{R}^q \backslash \tilde{N} \Rightarrow h^\psi \in \mathbb{R}^q \backslash \tilde{N} \subseteq \mathbb{R}^q \backslash N$$
$$\Rightarrow \hat{f}(h^\psi) = \lim_{n\to\infty} \hat{s}_n(h^\psi) = \lim_{n\to\infty} \hat{s}_n(h) = \hat{f}(h), \quad \text{by (11) and B.6}(c).$$

Thus $\hat{f}$ is symmetric on $\mathbb{R}^q \backslash \tilde{N}$ where $\tilde{N} \in \mathcal{N}_\sigma^{\mathrm{sym}}$. Thus (c). ■

Finally, let us note that the restriction (B.2) holds in the cases:

$$\mathcal{H} = \mathbb{R} \quad \& \quad \sigma \in \mathrm{CA}(\mathcal{D}_q, \mathbb{R}_{0+}), \quad \text{with} \quad c = 1,$$
$$\mathcal{H} = \mathcal{L}_2 \quad \& \quad \sigma = \eta_q \text{ (definition 9.6)} \quad \text{with} \quad c = \sqrt{q!}.$$

We have:

B.9. Lemma. (a) *Let* $\sigma \in \mathrm{CA}(\mathcal{D}_q, \mathbb{R}_{0+})$, $f \in L_{1,\sigma}$ *and* $f(\cdot) \geqslant 0$ *on* $\mathbb{R}^q$. *Then* $|f|_{1,\sigma} = 1 \cdot |\mathbb{E}_\sigma(f)|$.

(b) *Let* $q \in \mathbb{N}_+$, $f \in \mathcal{P}_{1,\eta_q}$ *and* $f(\cdot) \geqslant 0$ *on* $\mathbb{R}^q$. *Then* $|f|_{1,\eta_q} \leqslant \sqrt{q!}\, |\mathbb{E}_{\eta_q}(f)|_{\mathcal{L}_2}$.

Proof. (a) This is utterly obvious since now $\sigma \subseteq |\bar{\sigma}|$.

(b) Since $f(\cdot) \geqslant 0$ on $\mathbb{R}^q$, we see from 11.4(a) and 10.8 that

$$|f|_{1,\eta_q} \leqslant \sqrt{q!}\, |f|_{2,\ell_q} \leqslant \sqrt{q!}\, |\mathbb{E}_{\eta_p}(f)|.$$

■

Taking $f = \chi_D$, $D \in \mathcal{D}_q$, the restraint (B.2) reduces to $s_\sigma(D) \leqslant c|\sigma(D)|_{\mathcal{H}}$. For $\mathcal{H} = \mathbb{R}$, this further simplifies to

$$|\sigma|(D) \leqslant c|\sigma(D)|.$$

This inequality is violated by the simplest $\mathbb{R}$-valued measures, for instance, by $\sigma(D) = \sum_{k\in\mathbb{N}_+\cap D}(-1)^k$, $D \in \mathcal{D}_1$, for which $|\sigma|(D) = \#(\mathbb{N}_+ \cap D)$, and $D = \{1, 2\}$ provides a counter-example.

It is thus clear that most measures, scalar and vector, violate the restraint (B.2). However, as B.9 shows, the measure η_q has the demanded 'non-negativity' typified in the equality § 1(3). For the measure η_q, the use of the implication (1) $\Rightarrow$ (2) enables us to make a working principle out of the heuristic rule given in 12.8, and accordingly plays an important role in the integration theory of Wiener's measure ξ_p.

Appendix C. The ratio of finite sets of positive integers

Let B be a finite set of positive integers and $A \subseteq B$. To study the relations between the (binary-celled) partitions in the classes Π_A and Π_B, when A and B have even cardinality, cf. (1.16), we need to introduce a binary operation $|$, whereby $A|B$ is another finite set of positive integers.

C.1. *Definition.* Let (i) $\emptyset \neq A \subseteq B \subseteq [1, p]$,

$$\text{(ii)}\qquad B = \{b_1, b_2, \ldots, b_\beta\}, \quad \text{where} \quad 1 \leqslant b_1 < b_2 < \cdots < b_\beta \leqslant p,$$
$$A = \{b_{i_1}, b_{i_2}, \ldots, b_{i_\alpha}\}, \quad \text{where} \quad 1 \leqslant i_1 < i_2 < \cdots < i_\alpha \leqslant \beta.$$

Then we define the *ratio* $A|B$ to be the set $A|B := \{i_1, i_2, \ldots, i_\alpha\}$. For completeness we let $\emptyset|B := \emptyset$.

As the cancellation in the equality in C.3 below indicates, we should think of $|$ as a *division*, and of the set $A|B$ as the *ratio* of A over B. Obviously $A|B \subseteq [1, \#B]$ and $\#(A|B) = \#A$. It is good to also consider the ratio $(B\backslash A)|B$. We obviously have the following lemma:

C.2. Lemma. *Let $\emptyset \neq A \subseteq B \subseteq [1, p]$ & $\alpha := \#A$, $\beta := \#B$. Then*

(a) $A|B$, $(B\backslash A)|B \subseteq [1, \beta]$, $\#(A|B) = \alpha$, $\#\{(B\backslash A)|B\} = \#(B\backslash A) = \beta - \alpha$;

(b) $(B\backslash A)|B = [1, \beta]\backslash(A|B)$, $(A|B) = [1, \beta]\backslash\{(B\backslash A)|B\}$;

(c) in case $A = B$, $A|B = [1, \beta]$ & $(B\backslash A) = \emptyset$.

Example. Let $p = 12$, $A = \{2, 6, 7, 10, 12\}$ & $B = \{2, 5, 6, 7, 8, 9, 10, 12\}$. Then $\alpha = 5$, $\beta = 8$, $A|B = \{1, 3, 4, 7, 8\}$ and $(B\backslash A)|B = \{2, 5, 6\}$.

C.3. Lemma. *Let* (i) $\emptyset \neq A \subseteq B \subseteq [1, p]$ *and* (ii) $P \in \mathcal{P}_p$. *Then* $(P_B)_{A|B} = P_A$, *the notation being as in 1.9 and (1.34).*

Proof. Let as in C.1,

$$(1)\qquad B = \{b_1, b_2, \ldots, b_\beta\} \quad \text{where} \quad 1 \leqslant b_1 < b_2 < \cdots < b_\beta \leqslant p,$$

$$(2)\qquad A = \{b_{i_1}, b_{i_2}, \ldots, b_{i_\alpha}\} \quad \text{where} \quad 1 \leqslant i_1 < i_2 < \cdots < i_\alpha \leqslant \beta,$$

and let $P = P^1 \times P^2 \times \cdots \times P^p$, $P^k \in \mathcal{P}_1$. Then by (1) and 1.35(b),

$$P_B = P^{b_1} \times P^{b_2} \times \cdots \times P^{b_\beta} = Q^1 \times Q^2 \times \cdots \times Q^\beta, \quad \text{say.}$$

Since $A|B := \{i_1, i_2, \ldots, i_\alpha\}$, therefore,

$$\begin{aligned}(P_B)_{A|B} &= (Q^1 \times Q^2 \times \cdots \times Q^\beta)_{A|B} = Q^{i_1} \times Q^{i_2} \times \cdots \times Q^{i_\alpha} \\ &= P^{b_{i_1}} \times P^{b_{i_2}} \times \cdots \times P^{b_{i_\alpha}} \quad \text{since } Q^k := P^{b_k} \\ &=: P_A, \quad \text{by (2) \& 1.35}(b).\end{aligned}$$

■

Let $B \subseteq [1,p]$ and $\#B = \beta$. We claim that every subset J of $[1,\beta]$ is a ratio $(A|B)$ for some subset A of B, and that this A is uniquely determined by B and J, thus:

C.4. Triviality. *Let $p \in \mathbb{N}_+$ and $\emptyset \neq B \subseteq [1,p]$. Then*
(a) $2^{[1,\#B]} = \{A|B : A \subseteq B\}$;
(b) *given* $J \subseteq [1,\#B]$, $\exists$ *exactly one* $A \subseteq B \ni J = A|B$;
(c) $\forall \alpha \in [1,\#B]$, $\{J : J \subseteq [1,\#B]\ \&\ \#J = \alpha\} = \{A|B : A \subseteq B\ \&\ \#A = \alpha\}$.

Proof. (a) By C.1(ii), $A|B \subseteq [1,\#B]$. Hence RHS(a) $\subseteq$ LHS(a). Next let $\beta := \#B$ and take a member J of the family on the LHS(a), say $J = \{j_1, j_2, \ldots, j_\alpha\}$. Suppose that $B = \{b_1, b_2, \ldots, b_\beta\}$. Then obviously $\alpha \leqslant \beta$, and taking $A = \{b_{j_j}, \ldots, b_{j_\beta}\}$, we have by C.1, $J = A|B$. Thus LHS(a) $\subseteq$ RHS(a). Thus (a).

(b) Let $J = \{j_1, \ldots, j_\alpha\} \subseteq [1,\beta]$. Then by ($a$), $\exists A \subseteq B$ such that $J = A|B$. Let the members of B and A be as in the previous paragraph. Now suppose $\exists \hat{A} \subseteq B \ni \hat{A}|B = J$. Since $\hat{A} \subseteq B$, therefore $\exists n \in [1,\beta]$ and $\exists i_1, \ldots, i_n$ such that $\hat{A} = \{b_{i_1}, b_{i_2}, \ldots, b_{i_n}\}$ where $1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant \beta$. Thus

$$J = \hat{A}|B = \{i_1, \ldots, i_n\}.$$

Since $J = \{j_1, \ldots, j_\alpha\}$, it follows that $n = \alpha$ and $i_1 = j_1, \ldots, i_\alpha = j_\alpha$, and therefore,

$$\hat{A} = \{b_{i_1}, b_{i_2}, \ldots, b_{i_n}\} = \{b_{j_1}, b_{j_2}, \ldots, b_{j_\alpha}\} = A.$$

Thus (b).

(c) Let $J \subseteq [1,\beta]$. Then by (b) $\exists$ exactly one $A \subseteq B$ such that $J = A|B$. Now fix $\alpha \in [1,\beta]$. Then to prove (b) we have only to show that $\#J = \alpha \iff \#A = \alpha$. But this follows from C.2(a), since $\#J = \#(A|B) = \#A$. Thus (c). ■

The next result comes in handy in establishing a connection between the canonical coefficients γ^p_k, γ^p_j and γ^{p-2j}_{k-j} where $0 \leqslant j \leqslant k \leqslant [p/2]$, cf. 15.3.

C.5. Lemma. *Let* (i) $p \in \mathbb{N}_+$ *and* f *be a function on* $2^{[1,p]}$, (ii) $C \subseteq [1,p]$ *and* $c := \#C$. *Then*

$$\forall d \in \mathbb{N}_+ \ni c \leqslant d \leqslant p, \qquad \sum_{\substack{J \subseteq [1,p-c] \\ \#J = d-c}} f(J) = \sum_{\substack{C \subseteq D \subseteq [1,p] \\ \#D = d}} f\{(D \backslash C)|C'\},$$

where $C' := [1,p] \backslash C$.

Proof. Let $d \in \mathbb{N}_+$ and $c \leqslant d \leqslant p$. It will suffice to show that

(I) $\{J : J \subseteq [1,p-c]\ \&\ \#J = d-c\} = \{(D\backslash C)|C' : C \subseteq D \subseteq [1,p]\ \&\ \#D = d\}$.

For by (I) the two summations of the values of f in the lemma are over the same subfamily of $2^{[1,p]}$, and are therefore equal.

Proof of (I). Since $\#C' = p - c$ and $d - c \in [1, p-c]$, therefore by C.4(c),

(1) $$\text{LHS(I)} = \{A|C' : A \subseteq C'\ \&\ \#A = d - c\}.$$

Now let $A \subseteq C'$ and $A|C' \in \mathrm{RHS}(1)$. Then $A \subseteq C' \| C$ and $\#A = d - c$. Hence

$$D := A \cup C \subseteq [1,p] \quad \& \quad \#D = \#A + \#C = d.$$

Since trivially $A = D \backslash C$, it follows that

(2) $$\mathrm{RHS}(1) = \{(D\backslash C)|C' : C \subseteq D \subseteq [1,p] \quad \& \quad \#D = d\} = \mathrm{RHS(I)}.$$

Combining (1) and (2) we get (I). ■

So far binary partitions have not entered into this appendix. Now let $A \subseteq B \subseteq [1,p]$ be such that $\#(B\backslash A) = 2r$ is even. Then, by C.2(a), $\#\{(B\backslash A)|B\} = 2r$ also, and we can consider the classes of partitions $\Pi_{B\backslash A}$ and $\Pi_{(B\backslash A)|B}$. These classes have the same cardinality α_{2r}, cf. (1.17), and each partition in them has r cells. A one–one correspondence on $\Pi_{B\backslash A}$ to $\Pi_{(B\backslash A)|B}$ can be set up in many different ways. Now let $P \in \mathcal{P}_p$. Then since $B\backslash A \subseteq [1,p]$ and $\#(B\backslash A) = 2r$, it follows from (3.10) that

$$\forall \pi \in \Pi_{B\backslash A}, \qquad P(\pi) := \bigtimes_{\Delta \in \pi} P(\Delta) \in \mathcal{P}_r.$$

Also, $P_B \in \mathcal{P}_{\#B}$, $(B\backslash A)|B \subseteq B$ and $\#\{(B\backslash A)|B\} = 2r$. Hence, again by (3.10),

$$\forall \bar\pi \in \Pi_{(B\backslash A)|B}, \qquad P_B(\bar\pi) := \bigtimes_{\Delta \in \bar\pi} P_B(\Delta) \in \mathcal{P}_r.$$

We proceed to show that *corresponding to every* $\pi \in \Pi_{B\backslash A}$, *there is a unique* $\bar\pi \in \Pi_{(B\backslash A)|B}$ *such that* $\forall P \in \mathcal{P}_p$, $P_B(\bar\pi) = P(\pi)$.

Let $r \in [1,[p/2]]$ and $\pi \in \Pi_r^p$. We shall apply the considerations of the last paragraph, taking $A = M_\pi := {}^*\pi \cup \pi^*$, and taking B to be such that $M_\pi \subseteq B \subseteq [1,p]$. We wish to associate with π, a partition $\bar\pi$ in $\Pi_r^{\#B}$ such that $M_{\bar\pi} = M_\pi|B$. We have the following result:

C.6. Triviality. *Let* (i) $p \in \mathbb{N}_+$ & $r \in [1,[p/2]]$,
(ii) $\pi = \{\Delta_1, \ldots, \Delta_r\} \in \Pi_r^p$,
(iii) $M_\pi \subseteq B := \{b_1, \ldots, b_\beta\} \subseteq [1,p]$, $b_1 < \cdots < b_\beta$, $\beta \geqslant 2r$,
(iv) $\forall k \in [1,r]$, $\min \Delta_k = b_{i_k}$ & $\max \Delta_k = b_{j_k}$,
(v) $\bar\pi := \{\{i_1, j_1\}, \ldots, \{i_r, j_r\}\}$.
Then (a) $1 \leqslant i_1 < \cdots < i_r$; *each* $\{i_k, j_k\} \subseteq [1,\beta]$ & $\bar\pi \in \Pi_r^\beta$;
(b) $M_{\bar\pi} = M_\pi|B$;
(c) $\forall P \in \mathcal{P}_p$, $P_B(\bar\pi) = P(\pi)$, *where* P_B *is as in (1.34).*

Proof. By (iv) and (v),

(1) $$b_{i_k} = \min \Delta_k < \min \Delta_{k+1} = b_{i_{k+1}},$$

and by (iii)

(2) $$b_{j_k} = \max \Delta_k \leqslant \max B = b_\beta.$$

Since by (iii), $(b_1, b_2, \ldots, b_\beta)$ is increasing, it follows from (1) and (2) that

$$\forall k \in [1,r], \qquad i_k < i_{k+1} \quad \& \quad j_k \leqslant \beta.$$

From this and (v), we have (a).

(b) By (iv),

$$M_\pi = \bigcup_{k=1}^r \Delta_k = \{b_{i_1}, b_{j_1}, \ldots, b_{i_r}, b_{j_r}\} \subseteq \{b_1, \ldots, b_\beta\} = B.$$

Hence by (v) and the definition C.1 of the division |,

$$M_{\bar\pi} = \{i_1, j_1, \dots, i_r, j_r\} = M_\pi | B.$$

Thus (b).

(c) Let $P = P^1 \times \cdots \times P^p \in \mathcal{P}_p$. Then by (ii) and (iii),

(3) $$Q := P_B = P^{b_1} \times \cdots \times P^{b_\beta} \in \mathcal{P}_\beta.$$

Now by C.2, $M_\pi|B \subseteq [1,\beta]$, and by ($b$), $M_{\bar\pi} = M_\pi|B$. Hence by (3), (3.10) and (iv),

$$\begin{aligned} P_B(\bar\pi) = Q(\bar\pi) &= \bigtimes_{\bar\Delta\in\bar\pi} Q(\bar\Delta) = \bigtimes_{k=1}^{r} (Q^{i_k} \cap Q^{j_k}) \\ &= \bigtimes_{k=1}^{r} (P^{b_{i_k}} \times Q^{b_{j_k}}), \quad \text{since } Q^\nu = P^{b_\nu}, \text{ by (3)} \\ &= \bigtimes_{\Delta\in\pi} P(\Delta) = P(\pi), \quad \text{by (iv).} \end{aligned}$$

Thus (c). ■

C.7. *Definition.* Let $p \in \mathbb{N}_+$, $r \in [1,[p/2]]$ & $\pi \in \Pi_r^p$. Then $\forall B$ such that $M_\pi \subseteq B \subseteq [1,p]$, the (unique) partition $\bar\pi \in \Pi_r^{\#B}$ given in C.6(v) is called the *canonical* $M_\pi|B$ *associate* of π. Thus, if

$$B = \{b_1,\dots,b_\beta\} \quad \& \quad b_1 < \cdots < b_\beta \quad \& \quad \pi = \{\{b_{i_1}, b_{j_1}\},\dots,\{b_{i_r}, b_{j_r}\}\},$$

then $\bar\pi := \{\{i_1, j_1\},\dots,\{i_r, j_r\}\}$.

Note. Since $M_{\bar\pi} = M_\pi|B$, by C.6(b), therefore $\bar\pi \in \Pi_{M_\pi|B}$, and by C.6($c$), $P_B(\bar\pi) = P(\pi)$.

Let $A \subseteq B \subseteq [1,p]$ and $\#(B\backslash A)$ be even. We shall now show that every partition in $\Pi_{(B\backslash A)|B}$ is the canonical associate of a partition in $\Pi_{B\backslash A}$, i.e. that the canonical correspondence is 'onto'.

C.8. Proposition. *Let* (i) $p \in \mathbb{N}_+$ *&* $r \in [1,[p/2]]$, (ii) $A \subseteq B \subseteq [1,p]$ *and* $\#(B\backslash A)$ *be even,* (iii) $\pi \in \Pi_{B\backslash A}$ *and* $\bar\pi \in \Pi_{(B\backslash A)|B}$ *be the* $M_\pi|B$ *canonical associate of* π. *Then*

(a) $\Pi_{(B\backslash A)|B} = \{\bar\pi : \pi \in \Pi_{B\backslash A}\} = \Pi_{(B\backslash A)|A'}$, *where* $A' := [1,p]\backslash A$;
thus the canonical correspondence $\pi \to \bar\pi$ *is one–one on* $\Pi_{B\backslash A}$ *onto* $\Pi_{(B\backslash A)|B}$;

(b) $\forall P \in \mathcal{P}_p \quad \& \quad \forall \pi \in \Pi_{B\backslash A}, \qquad P(\pi) = P_B(\bar\pi)$.

Proof. (a) Let $\pi \in \Pi_{B\backslash A}$. Then by C.6($b$), $M_{\bar\pi} = M_\pi|B = (B\backslash A)|B$, and therefore $\bar\pi \in \Pi_{(B\backslash A)|B}$. Thus $\pi \in \Pi_{B\backslash A} \Longrightarrow \bar\pi \in \Pi_{(B\backslash A)|B}$.

To show the reverse implication, let

(1) $$B = \{b_1, b_2,\dots,b_\beta\} \quad \text{where} \quad 1 \leqslant b_1 < b_2 < \cdots < b_\beta \leqslant p,$$

(2) $$B\backslash A = \{b_{i_1}, b_{i_2},\dots,b_{i_{2r}}\} \quad \text{where} \quad 1 \leqslant i_1 < i_2 < \cdots < i_{2r} \leqslant \beta.$$

Then by the definition of |,

(3) $$(B\backslash A)|B = \{i_1, i_2,\dots,i_{2r}\}.$$

Now let $\pi' \in \Pi_{(B\backslash A)|B}$. Then by (3),

(4) $$\pi' = \{\{i_{\mu_1}, i_{\nu_1}\},\{i_{\mu_2}, i_{\nu_2}\},\dots,\{i_{\mu_r}, i_{\nu_r}\}\},$$

where

(5) $\quad i_1 \leqslant i_{\mu_1} < i_{\mu_2} < \cdots < i_{\mu_r}, \quad \& \quad i_{\mu_1} < i_{\nu_1} \quad \& \quad \ldots \quad \& \quad i_{\mu_r} < i_{\nu_r} \leqslant i_{2r}.$

(6) $$\{i_{\mu_1}, i_{\nu_1}, \ldots, i_{\mu_r}, i_{\nu_r}\} = \{i_1, \ldots, i_r\} = (B\backslash A)|B.$$

Now define

(7) $$\pi := \{\{b_{i_{\mu_1}}, b_{i_{\nu_1}}\}, \{b_{i_{\mu_2}}, b_{i_{\nu_2}}\}, \ldots, \{b_{i_{\mu_r}}, b_{i_{\nu_r}}\}\}.$$

Then obviously from the inequalities in (1), (2) and (5),

$$b_{i_1} \leqslant b_{i_{\mu_1}} < b_{i_{\mu_2}} < \cdots < b_{i_{\mu_r}}, \quad \& \quad b_{i_{\mu_1}} < b_{i_{\nu_1}} \quad \& \quad \ldots \quad \& \quad b_{i_{\mu_r}} < b_{i_{\nu_r}} \leqslant b_{2r},$$

and by (6) and (3),

$$\{b_{i_{\mu_1}}, b_{i_{\nu_1}}, \ldots, b_{i_{\mu_r}}, b_{i_{\nu_r}}\} = \{b_{i_1}, \ldots, b_{i_r}\} = (B\backslash A)|B.$$

Hence by (7), $\pi \in \Pi_{(B\backslash A)|B}$.

It follows from (7) and C.6(v) that

(8) $$\bar{\pi} = \{\{i_{\mu_1}, i_{\nu_1}\}, \{i_{\mu_2}, i_{\nu_2}\}, \ldots, \{i_{\mu_r}, i_{\nu_r}\}\},$$

i.e. by (4), $\bar{\pi} = \pi'$. We have thus shown that $\forall \pi' \in \Pi_{(B\backslash A)|B}$, $\exists \pi \in \Pi_{B\backslash A}$ such that $\bar{\pi} = \pi'$. This establishes the implication: $\bar{\pi} \in \Pi_{(B\backslash A)|B} \Longrightarrow \pi \in \Pi_{B\backslash A}$. Thus we have

$$\pi \in \Pi_{B\backslash A} \quad \Longleftrightarrow \quad \bar{\pi} \in \Pi_{(B\backslash A)|B},$$

which establishes the first equality in (a).

As for the second equality in (a), note that $B\backslash A - A'\backslash B'$, and $B' \subseteq A' \subseteq [1, p]$. Hence, applying the first equality,

$$\Pi_{(B\backslash A)|A'} = \Pi_{(A'\backslash B')|A'} = \{\bar{\pi} : \pi \in \Pi_{A'\backslash B'}\} = \{\bar{\pi} : \pi \in \Pi_{B'\backslash A'}\}.$$

This proves (a)

(b) This just repeats C.6(c). ∎

Index of notation

Symbol	Location		
$:=$, $\forall$, $\exists$, $\#(A)$, χ_A, $\|$, $\perp$	1.1(a)	α_p	1.1(i)
LHS, RHS	1.1(a)	$\mathcal{D}^{\mathrm{loc}}$	(1.3), (A.1)
$\mathrm{Rstr.}_A f$, $\mathbb{F}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$	1.1(a), (b)	$\mathcal{L}_2$	(1.3)
$\mathbb{R}_+$, $\mathbb{N}_+$, $\mathbb{R}_{0+}$, $\mathbb{N}_{0+}$	1.1(b)	$\mathrm{CAOS}(\mathcal{D}, \mathcal{L}_2)$	1.6
$(a, b]$, $[a, b]$, $[m, n]$	1.1(c)	$\mathcal{D}_p$, $\mathcal{B}_p$, $\mathcal{P}_p$, $\mathcal{R}_p$,	1.9
$\mathbb{R}^p$, $\mathbb{R}^0$	1.1(d)	ℓ_p, $\|\ell_p\|$, $\bar{\mathcal{D}}_p$	1.9
$\mathcal{M}(\mathcal{F}, \mathcal{G})$	1.1(e)	$\mathcal{F}^{\mathrm{synm}}$	1.9
$\langle A \rangle$, cls, $\mathfrak{S}(A)$	1.1(f)	$\mathcal{R}_p$	(1.10)
$\mathrm{L}(X, Y)$, $\mathrm{CL}(X, Y)$	1.1(f)	ξ_p on $\mathcal{P}_p$	1.13
$\mathrm{FA}(\mathcal{R}, Y_0)$, $\mathrm{CA}(\mathcal{R}, Y_0)$	1.1(g)	Π_M, $\Pi_\emptyset$, π, ${}^*\pi$, π^*	1.16
$\mathcal{M}_\xi$, $\mathcal{S}_\xi$	1.1(g)	Π^p_k, M_π, M'_π	1.16
σ-alg$(\mathcal{F})$, σ-ring$(\mathcal{F})$, etc.	1.1(g)	$\times_{i=1}^{\eta} A_i$	1.30
$\mathbb{E}_\xi(f)$	1.1(h), (A.25), A.26	$\wp_M$	1.31
		$\wp_M(A)$, $\wp_M^{-1}(A)$	1.32
		$\mathrm{Perm}(p)$	(1.36)

t^{ϕ}, $A^{\phi^{-1}}$	1.37	$\mathbb{A}_{\xi}$	(8.13)
f^{ϕ}, $\tilde{f}$	1.39	$\eta_{p,q}$, $\zeta_{p,q}$	(9.1)
ξ_0, ℓ_0	3.1	η_p, ζ_p	9.6
$P(\Delta)$, $P(\pi)$	(3.10)	$\xi_{p,k}$	(9.15)
$a^n_{\pi}(P)$	(3.11)	J^{p+q}_1	(11.11)
$\Gamma^{pq}_k(P,Q)$	3.13	$\mathring{\Pi}^{p+q}_r$	11.15
I^p_{ij}, I^p_1, $\mathbb{R}^p_*$	(4.1)	$\theta^p_{\pi,h}(\tau)$, $f^p_{\pi}(\tau,h)$, $J^p_{\pi,h}$	12.2
S^p_{ϕ}	4.2(*b*)	$H^p_{\pi}(f)$	12.9(*d*)
$I(\pi,p)$	4.3	$H^p_k(f)$, $\mathcal{M}^p_k$	12.11
I^p_k	(4.5)	$f^p_k(\cdot)$, $M^p_k(f)$	12.11
$I^p_{\pi}(h)$	4.6	For $\pi \in \Pi^{p+q}_r$ & $i = 0, 1, 2$	14.5
$A^p_{\pi}(h)$	4.10	π_i, A_i, τ^i	14.5(*a*), (*b*)
$\lambda^p_{\pi}(D,h)$, $\gamma^p_k(D,h)$	4.13	h^1, h^2, $\hat{h}^1$, $\hat{h}^2$	14.5(*a*), (*b*)
$\|\lambda^p_{\pi}\|(\cdot,h)$, $\|\gamma^p_k\|(\cdot,h)$	4.17	$[\tau_0, h^1]$, $[\tau_0, h^2]$	(14.8)
$\Gamma^{pq}_k(D,E)$	5.1	$H_p(u,\sigma)$	(16.2)
ξ_p on $\mathcal{D}_p$	5.6	$h_p(f)$	(16.8)
$\rho \ll \mu$	(5.14)	$q_{\rho}(A)$, $s_{\rho}(A)$	A.3
$\xi^a_p(D)$, $\xi^b_p(D)$	(5.16)	$\mathcal{N}_{\rho}$	(A.5)
ϕ-distortion of π	6.4	$\mathcal{D}_{\rho}$	(A.6)
(ϕ,π)-permutation	6.4	$\|f\|_{1,x'\circ\rho}$, $\|f\|_{1,\rho}$	A.9
M-extension $\bar{\psi}_M$	6.6	$\mathcal{G}_{1,\rho}$	(A.10)
π-extension $\bar{\psi}_{\pi}$	6.6	$\mathcal{S}(\mathcal{F},\mathbb{R})$	(A.12)
π_k	(6.11)	$\mathcal{P}_{1,\rho}$	(A.14)
ϕ_{π}	6.12	$\mathbb{E}_{\rho}(f)$	(A.25),
$[h]^{j,k}_{\alpha,\beta}$	7.2		A.26
$0 \cdot A$, $1 \cdot A$, (A, a set)	7.4	$\mathcal{D}_{\rho}(f)$, $\nu_{\rho,f}(A)$	A.36
$\mathcal{L}^{\xi}_2$, $(\mathcal{L}^{\xi}_2)^+$, $(\mathcal{L}^{\xi}_2)^-$	(8.11)	$A\|B$	C.1

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